On the use of fractional derivative operators to describe viscoelastic damping in structural dynamics - FE formulation of sandwich beams and approximation of fractional derivatives by using the G^α scheme

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Dérivation fractionnaire en mécanique – Etat-de-l'art et applications

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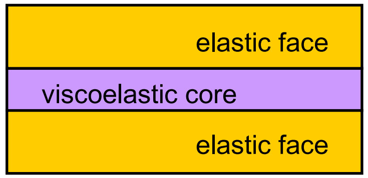
Objectives

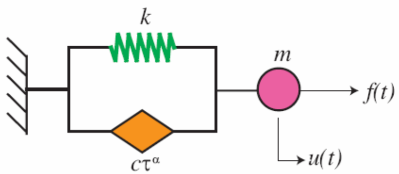
- To implement a fractional derivative model into a finite element code in structural dynamics

Sandwich beam

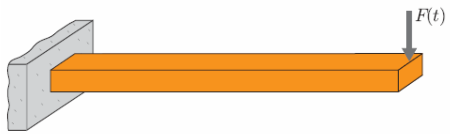


- To test a finite difference based on scheme to approximate fractional derivative operators in viscoelastic constitutive equations

Damped oscillator



Viscoelastic beam



Outline

I. Implementation of the fractional Zener model into a FE code – application to sandwich beams

- Constitutive equations – fractional Zener model
- Grünwald-Letnikov approximation
- FE formulation & algorithm
- Examples: validation & results

II.Testing the G^{α} scheme

- Outline
- Elementary tests
- Error estimates
- The damped oscillator problem
- Viscoelastic cantilever beam

I. Implementation of the fractional Zener model into a FE code – application to sandwich beams

Fractional derivative viscoelastic model

One-dimensional constitutive equation

$\Rightarrow$ Fractional derivative Zener Model (4 parameters)

R.L. Bagley and P.J. Torvik, AIAA (1983)

In frequency domain

$$E^*(\omega) = \frac{E_o + E_\infty (i\omega\tau)^\alpha}{1 + (i\omega\tau)^\alpha}$$

$$0 < \alpha < 1$$

$$E_\infty > E_o$$

$$\tau > 0$$

In time domain

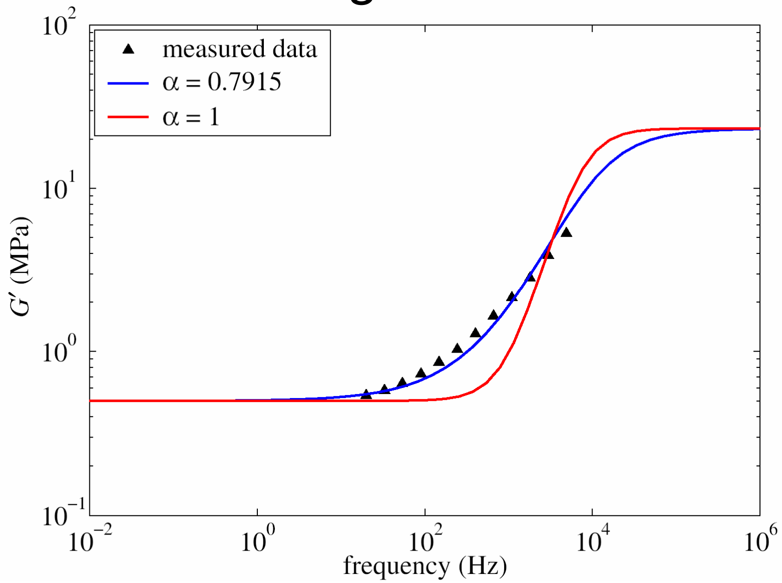
$$\boxed{\sigma(t) + \tau^\alpha D^\alpha \sigma(t) = E_o \varepsilon(t) + \tau^\alpha E_\infty D^\alpha \varepsilon(t)}$$

with $D^\alpha \sigma(t) = \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_0^t \dfrac{\sigma(s)}{(t-s)^\alpha} \mathrm{d}s$

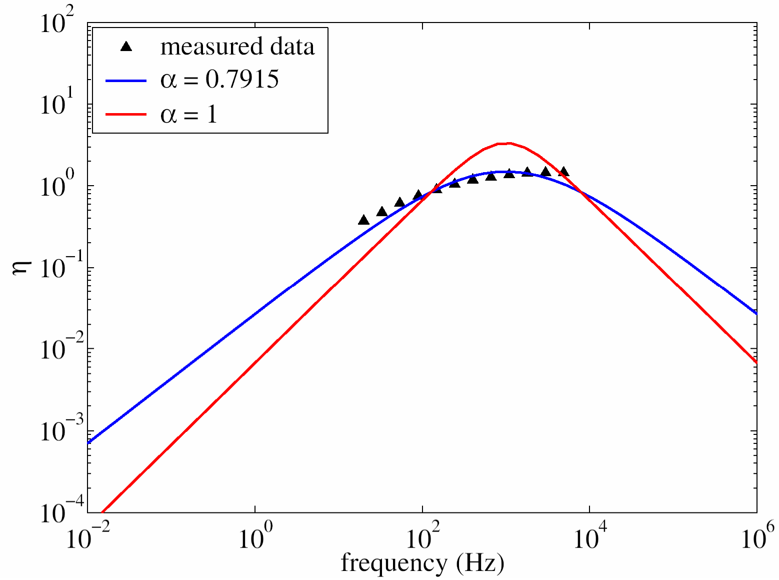
Master curves of the ISD112 (3M™) at 27°

Fractional model versus classical Zener model

Storage modulus



Loss factor



Identified model parameters:

$$E_o = 1.5 \text{ MPa}, \; E_\infty = 69.9 \text{ MPa}, \; \alpha = 0.7915, \; \tau = 1.4052 \times 10^{-2} \text{ ms}$$

Grünwald approximation & time discretization

$$\sigma(t) + \tau^\alpha D^\alpha \sigma(t) = E_o \varepsilon(t) + \tau^\alpha E_\infty D^\alpha \varepsilon(t)$$

Approximation for fractional derivatives

J. Padovan, Comp. Mech. (1987)

A. Schmidt and L. Gaul, Proc. 2nd ECCMR (2001)

$$\boxed{(D^\alpha f)_n \approx \frac{1}{\Delta t^\alpha} \sum_{j=0}^{N_t} A_{j+1}\, f_{n-j}}$$

with $$A_{j+1} = \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)}$$

Variable changing ⇨ Only one fractional derivative term

Discretizing…

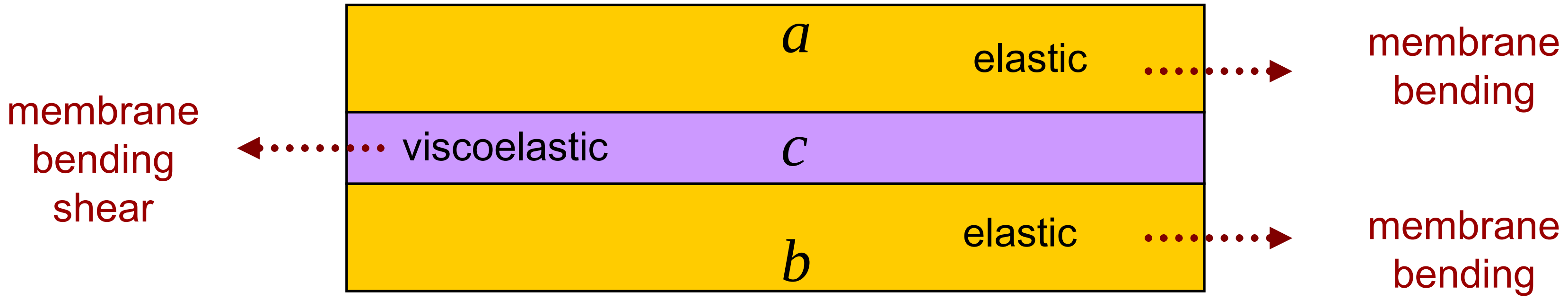
$$\boxed{\bar{\varepsilon}_i^{n+1} = (1-c)\frac{E_\infty - E_o}{E_\infty} \varepsilon_i^{n+1} - c \sum_{j=1}^{N_t} A_{j+1} \bar{\varepsilon}_i^{n+1-j}}$$

with $$c = \frac{\tau^\alpha}{\tau^\alpha + \Delta t^\alpha}$$

Finite element formulation

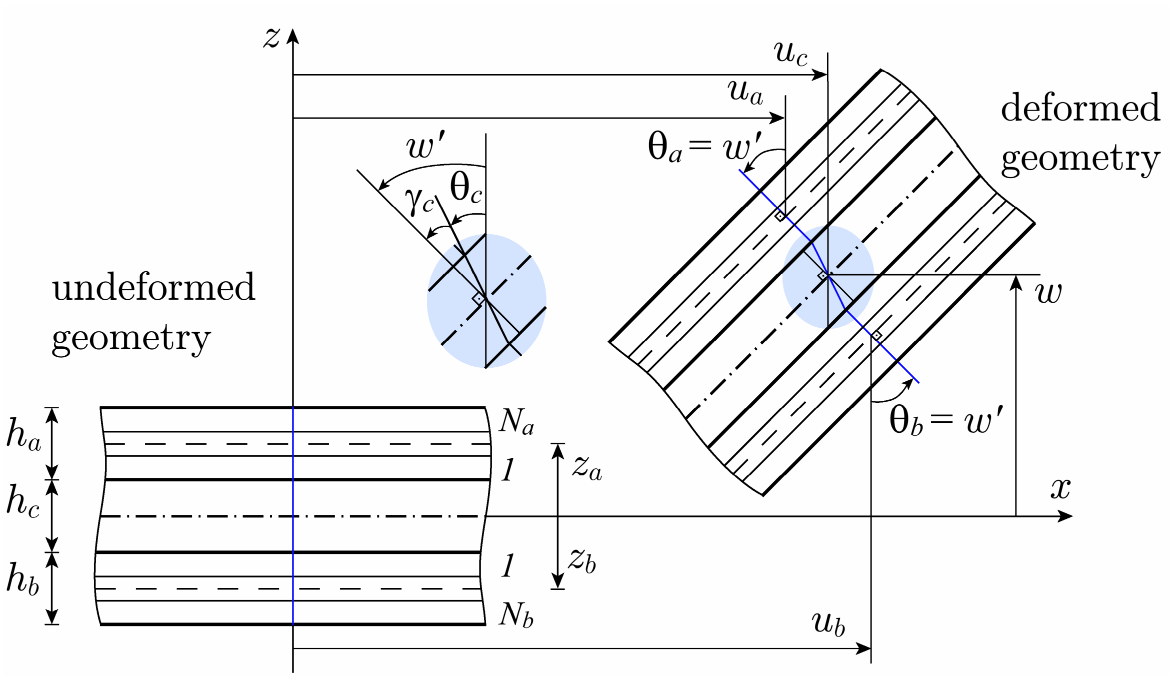
- Sandwich beam element
- Fractional model implementation
- Dynamic time integration

The structure consists on a three-layer beam
$\Rightarrow$ constrained layer damping treatment



Sandwich-beam kinematics

Mead and Markus, JSV (1969), Trindade, Benjeddou and Ohayon, IJNME (2001)



Constrained layer damping treatment

- Laminated elastic faces: Euler-Bernoulli beam
- Viscoelastic core: Timoshenko beam

Viscoelastic core

$$\int_{\Omega_c^e} \mathrm{B}_c^{\mathrm{T}} \boldsymbol{\sigma}_c^{n+1} \mathrm{d}\Omega = (\mathbf{K}_c^e + \bar{\mathbf{K}}_c^e)\mathbf{q}_{n+1}^e - \bar{\mathbf{F}}_{n+1}^e$$

modified stiffness matrix

$$\bar{\mathbf{K}}_c^e = c\frac{E_\infty - E_o}{E_o}\mathbf{K}_c^e$$

modified loading

$$\bar{\mathbf{F}}_e^{n+1} = -c\frac{E_\infty}{E_o}\mathbf{K}_e^c \sum_{j=1}^{N_t} A_{j+1}\bar{\mathbf{q}}_e^{n+1-j}$$

"Anelastic displacements" are evaluated as functions of their own time histories

$$\bar{\mathbf{q}}_e^{n+1} = (1-c)\frac{E_\infty - E_o}{E_\infty}\mathbf{q}_e^{n+1} - c\sum_{j=1}^{N_t} A_{j+1}\bar{\mathbf{q}}_e^{n+1-j}$$

Elementary dof vector $\mathbf{q}^e = [\bar{u}_1 \;\; w_1 \;\; w_1' \;\; \tilde{u}_1 | \bar{u}_2 \;\; w_2 \;\; w_2' \;\; \tilde{u}_2]^{\mathrm{T}}$

Equation of motion

$$\boxed{\mathbf{M}_e\ddot{\mathbf{q}}_e^{n+1} + (\mathbf{K}_e + \bar{\mathbf{K}}_e)\mathbf{q}_e^{n+1} = \mathbf{F}_e^{n+1} + \bar{\mathbf{F}}_e^{n+1}}$$

+ initial conditions

Remarks

- For an elastic material ($\tau = 0$), classical elastic stiffness matrix is obtained

$$\cancel{\bar{\mathbf{K}}_c^e = c\frac{E_\infty - E_o}{E_o}\mathbf{K}_c^e} \qquad c = 0$$

- For the Zener model ($\alpha = 1$), previous equations correspond to a backward Euler discretization of the constitutive equations

$$c = \frac{\tau}{\tau + \Delta t}$$

Dynamic time integration

$$\mathbf{M}\ddot{\mathbf{q}}_{n+1} + \mathbf{K}^*\mathbf{q}_{n+1} = \mathbf{F}_{n+1} + \bar{\mathbf{F}}_{n+1}$$

evaluated once

$$\boxed{\mathbf{K}^* = \mathbf{K} + \bar{\mathbf{K}}_c}$$

Newmark scheme

$$\beta = 1/4 \qquad \gamma = 1/2$$

$$\boxed{t_{n+1} = t_n + \Delta t}$$

$$\begin{aligned} \mathbf{q}_{n+1}^{pred} &= \mathbf{q}_n + \Delta t\dot{\mathbf{q}}_n + (0.5-\beta)\Delta t^2\ddot{\mathbf{q}}_n \\ \dot{\mathbf{q}}_{n+1}^{pred} &= \dot{\mathbf{q}}_n + (1-\gamma)\Delta t\ddot{\mathbf{q}}_n \end{aligned}$$

$$\boxed{\bar{\mathbf{F}}_{n+1} = -c\tfrac{E_\infty}{E_o}\mathbf{K}_c \sum_{j=1}^{N_t} A_{j+1}\bar{\mathbf{q}}_{n+1-j}}$$

$$\begin{aligned} \mathbf{q}_{n+1} &= \mathbf{q}_{n+1}^{pred} + \beta\Delta t^2\ddot{\mathbf{q}}_{n+1} \\ \dot{\mathbf{q}}_{n+1} &= \dot{\mathbf{q}}_{n+1}^{pred} + \gamma\Delta t\ddot{\mathbf{q}}_{n+1} \end{aligned}$$

$$\boxed{\mathbf{R}_{n+1} = \mathbf{F}_{n+1} + \bar{\mathbf{F}}_{n+1} - \mathbf{K}^*\mathbf{q}_{n+1}^{pred}}$$

$$\boxed{(\mathbf{M} + \beta\Delta t^2\mathbf{K}^*)\ddot{\mathbf{q}}_{n+1} = \mathbf{R}_{n+1}}$$

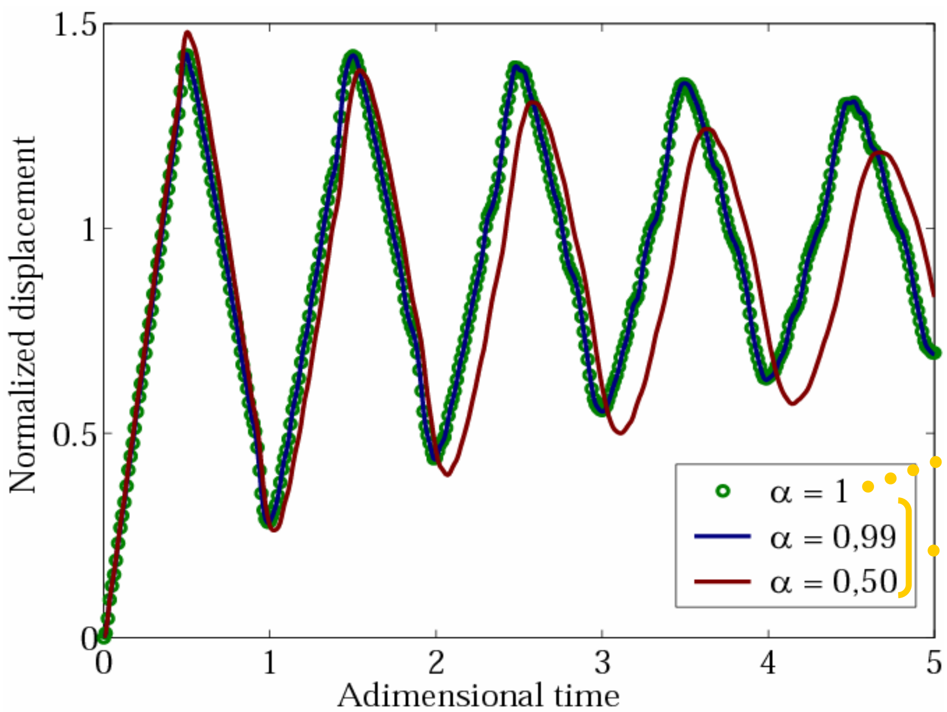
storage of "anelastic displacements"

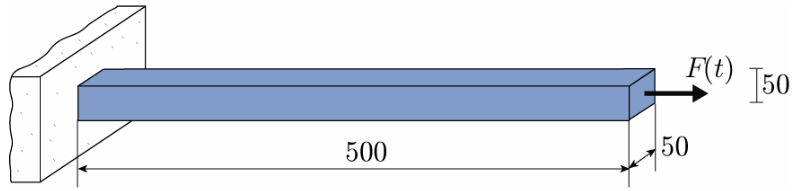
$$\boxed{\bar{\mathbf{q}}_{n+1}^e = (1-c)\frac{E_\infty - E_o}{E_\infty}\mathbf{q}_{n+1}^e - c\sum_{j=1}^{N_t} A_{j+1}\bar{\mathbf{q}}_{n+1-j}^e}$$

Validation – Example 1

Viscoelastic fixed-free bar subjected to unit step load

M. Enelund et B.L. Josefson, AIAA Journal (1997)





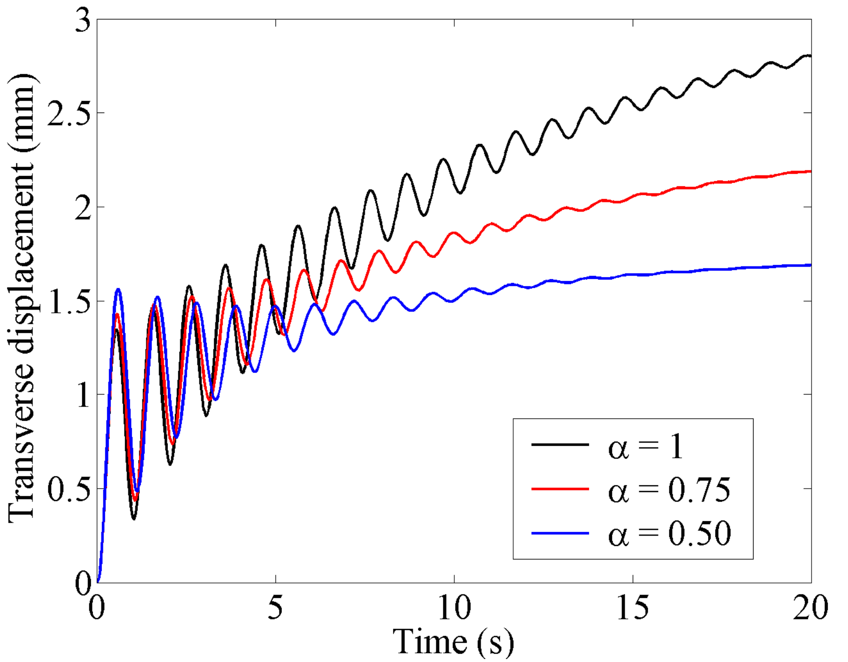
Classical algorithm
Taylor et al., IJNME (1970)

Fractional derivatives model

Validation – Example 2

Viscoelastic simply-supported Timoshenko beam subjected to a uniform step function load

T.M. Chen, IJNME (1995)

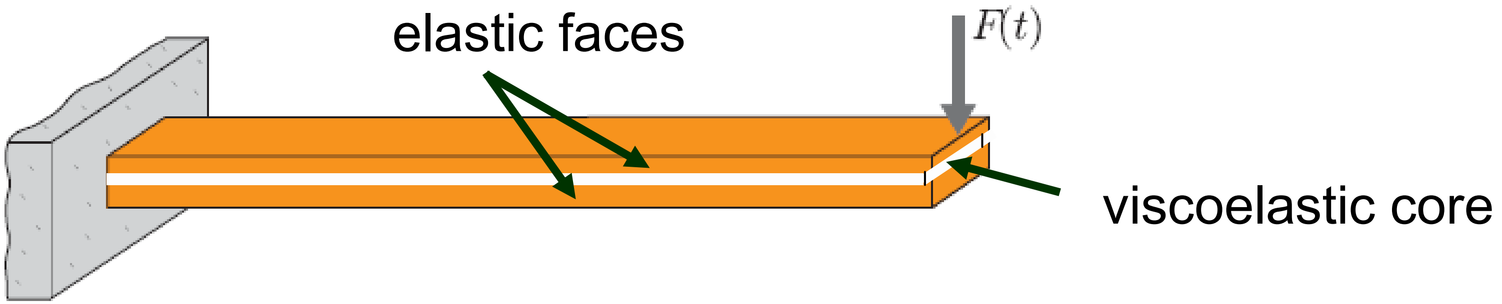


Validation for $\alpha = 1$

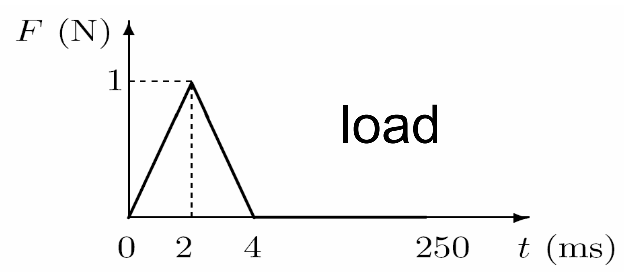
good agreement with the “hybrid Laplace transform/FE method”

Results

Sandwich beam subjected to a triangular pulse



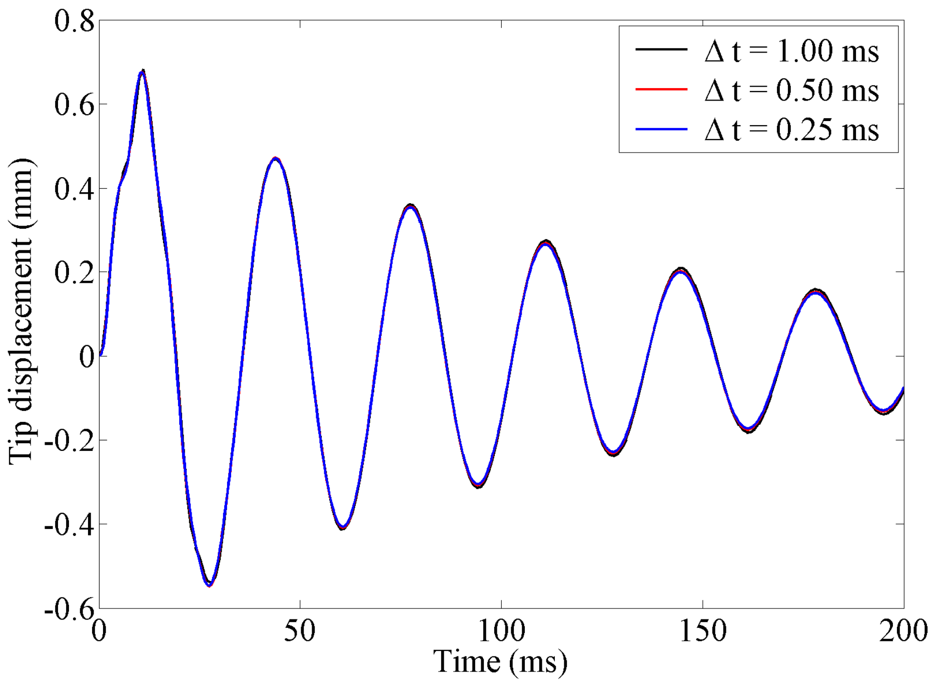
length = 280 mm
width = 25 mm
thickness:
• core = 0.2 mm
• skin = 1.5 mm



Results

Influence of the time step and truncation

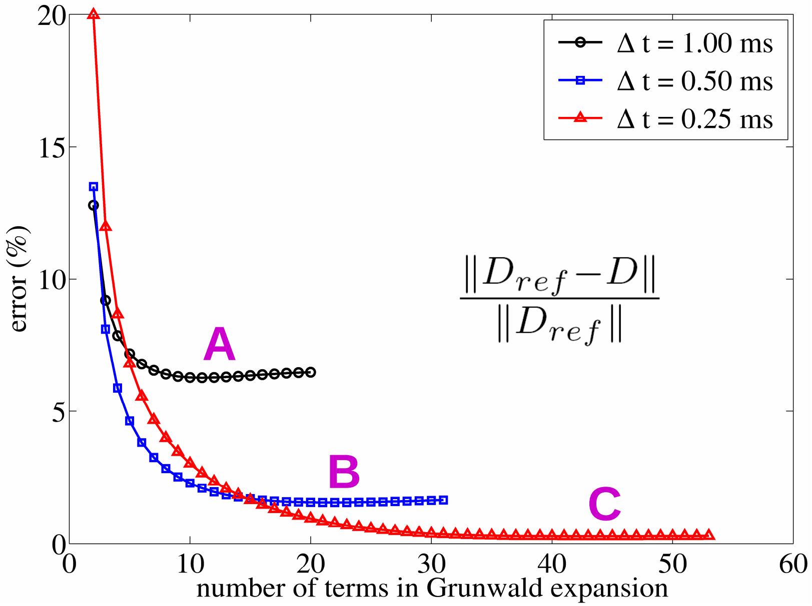
Calculations performed with whole history

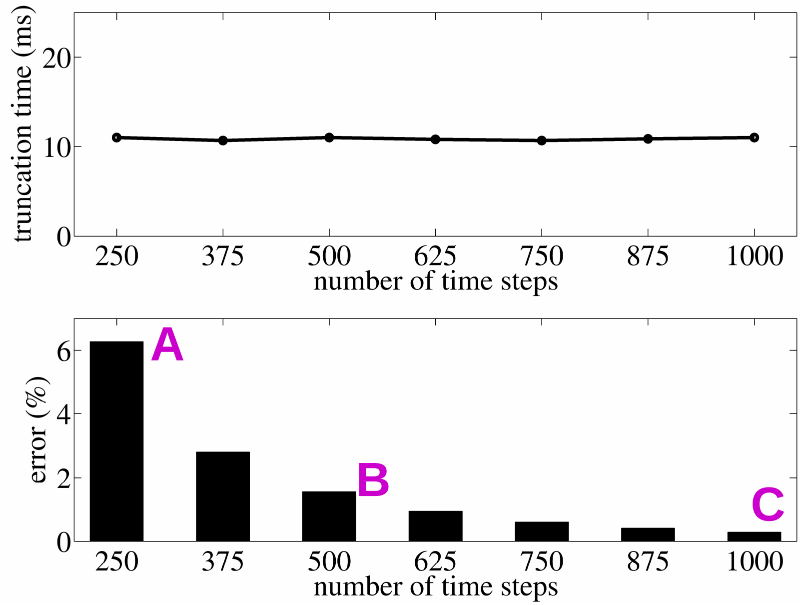


Remark: Energy balance is strictly zero.

Results

Influence of the time step and truncation





D_{ref} Very accurate solution with a fine time discretization:

$N_{step} = 2500 \quad N_t = 2500$

Existence of a truncation time where the recent history must be stored.

II. Testing the G^{α} scheme

How to obtain a fractional derivative operator

Euler backward operator

$$\mathrm{E} = \frac{1}{\Delta t}(I - \delta^-)$$

Grünwald-Letnikov operator

$$\mathrm{GL} = \mathrm{E}^\alpha = \frac{1}{\Delta t^\alpha}(I - \delta^-)^\alpha$$

Gear operator

$$\mathrm{G} = \frac{1}{\Delta t}\left[\frac{3}{2}I - 2\delta^- + \frac{1}{2}(\delta^-)^2\right]$$

G^α operator

$$\mathrm{G}^\alpha = \frac{1}{\Delta t^\alpha}\left(\frac{3}{2}\right)^\alpha \left[I - \frac{4}{3}\delta^- + \frac{1}{3}(\delta^-)^2\right]^\alpha$$

The G$^\alpha$ scheme

The α-derivative of u at time t^n can be approximated by:

$$(\mathrm{G}^\alpha u)^n = \frac{1}{\Delta t^\alpha}\left(\frac{3}{2}\right)^\alpha \sum_{j=0}^{\infty} g_{j+1} u^{n-j}$$

First five coefficients g_{j+1}

j	$\alpha = 1/3$	$\alpha = 1/2$	$\alpha = 3/4$
0	1	1	1
1	$-\frac{4}{9}$	$-\frac{2}{3}$	-1
2	$-\frac{7}{81}$	$-\frac{1}{18}$	$\frac{1}{12}$
3	$-\frac{104}{2187}$	$-\frac{1}{27}$	$-\frac{1}{108}$
4	$-\frac{643}{19683}$	$-\frac{17}{648}$	$-\frac{1}{96}$
5	$-\frac{4348}{177147}$	$-\frac{19}{972}$	$-\frac{7}{864}$

Elementary tests

Consider the following power function

$$u(t) = t^{\nu}, \quad \nu \geq 0$$

We search to calculate its α-derivative by using an exact solution and an approximated one with the G^{α}-scheme, such that:

Exact solution

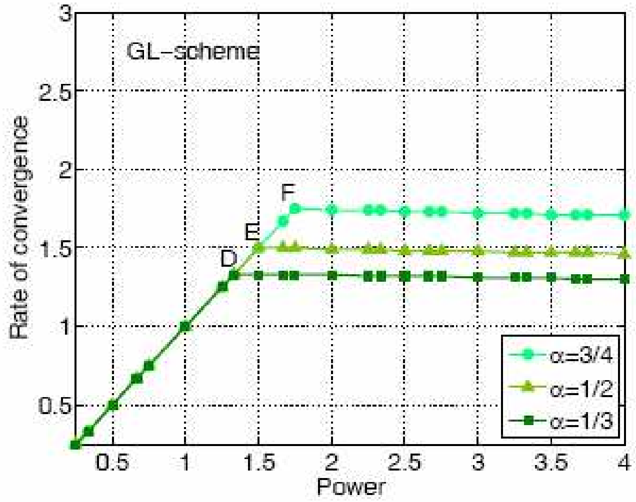
$$\mathcal{D}^{\alpha} u(t) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\alpha)} t^{\nu-\alpha}$$

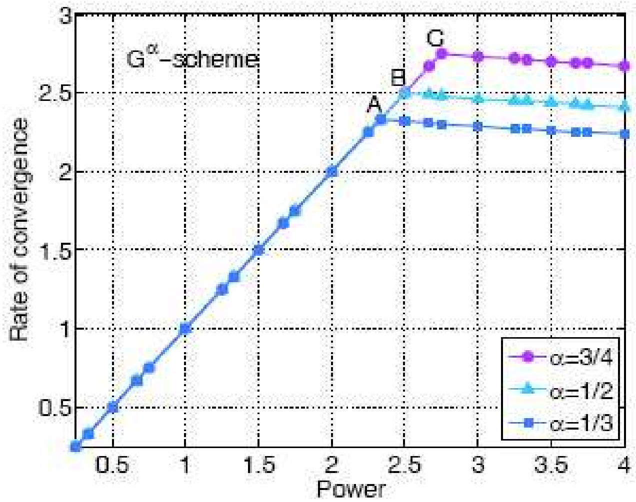
Approximated solution

$$(G^{\alpha} u)^{n} = \frac{1}{\Delta t^{\alpha}} \left(\frac{3}{2}\right)^{\alpha} \sum_{j=0}^{\infty} g_{j+1} u^{n-j}$$

How accurate is the G^{α}-scheme for this case?

Error estimates in L^∞ norm





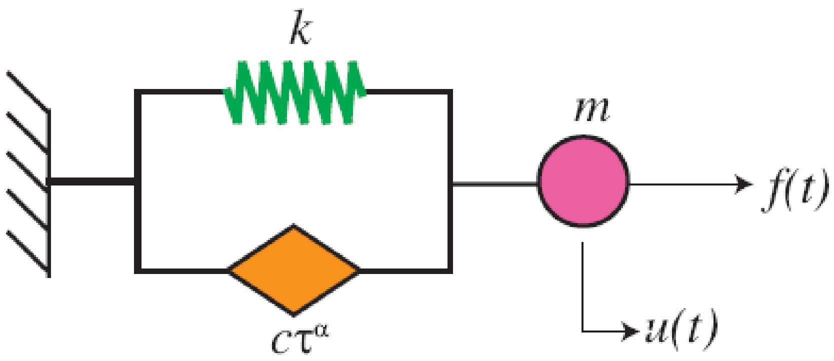
	$\alpha = 1/3$		$\alpha = 1/2$		$\alpha = 3/4$	
Power	G^α	GL	G^α	GL	G^α	GL
1/4	0.25	0.25	0.25	0.25	0.25	0.25
1/3	0.33	0.33	0.33	0.33	0.33	0.33
1/2	0.50	0.50	0.50	0.50	0.50	0.50
2/3	0.67	0.67	0.67	0.67	0.67	0.67
3/4	0.75	0.75	0.75	0.75	0.75	0.75
1	1.00	1.00	1.00	1.00	1.00	1.00
5/4	1.25	1.25	1.25	1.25	1.25	1.25
4/3	1.33	1.33 D	1.33	1.33	1.33	1.33
3/2	1.50	1.33	1.50	1.50 E	1.50	1.50
5/3	1.67	1.33	1.67	1.50	1.67	1.67
7/4	1.75	1.33	1.75	1.50	1.75	1.75 F
2	2.00	1.33	2.00	1.49	2.00	1.74
9/4	2.25	1.32	2.25	1.49	2.25	1.74
7/3	2.33 A	1.32	2.33	1.49	2.33	1.74
5/2	2.32	1.32	2.50 B	1.48	2.50	1.73
8/3	2.31	1.32	2.49	1.48	2.67	1.73
11/4	2.30	1.32	2.48	1.48	2.75 C	1.73
3	2.29	1.31	2.46	1.48	2.73	1.72
13/4	2.27	1.31	2.45	1.47	2.72	1.72
10/3	2.27	1.31	2.45	1.47	2.71	1.72
7/2	2.26	1.31	2.44	1.47	2.70	1.71
11/3	2.25	1.30	2.43	1.47	2.69	1.71
15/4	2.25	1.30	2.42	1.47	2.69	1.71
4	2.24	1.30	2.41	1.46	2.67	1.71

G^α: $\alpha + 2$ GL: $\alpha + 1$

The damped oscillator problem

The problem is given such that

$$\begin{cases} m\ddot{u} + c\tau^{\alpha}\mathcal{D}^{\alpha}u + ku = f, \quad t > 0 \\ u(0) = \dot{u}(0) = 0 \end{cases}$$



Algorithm

We search u^{n+1} such that

$$m\ddot{u}^{n+1} + (k+\kappa)u^{n+1} = f^{n+1} + \phi^{n+1}$$

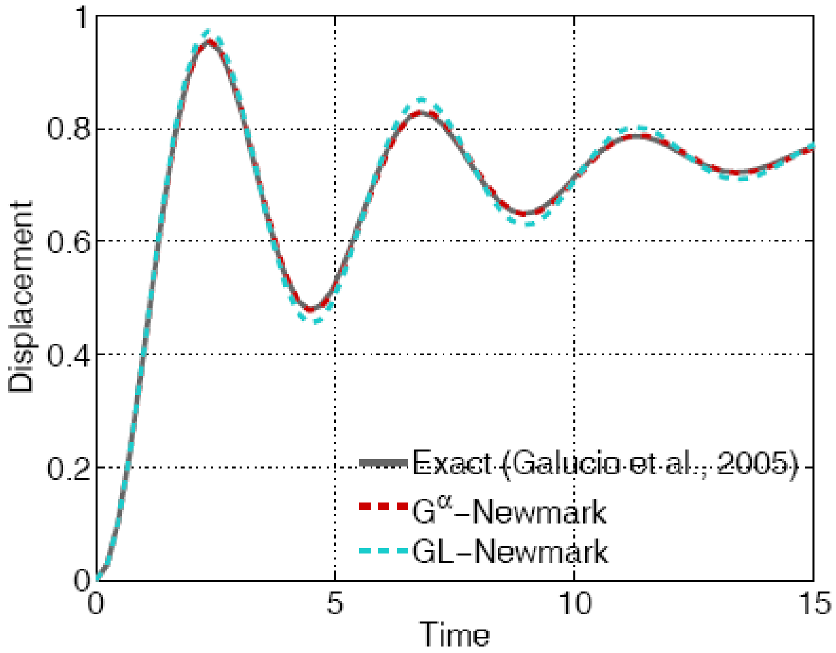
where $\kappa = \frac{c\tau^\alpha}{\Delta t^\alpha}\left(\frac{3}{2}\right)^\alpha$ and $\phi^{n+1} = -\frac{c\tau^\alpha}{\Delta t^\alpha}\left(\frac{3}{2}\right)^\alpha \sum_{k=1}^{N} g_{k+1}u^{n+1-k}$

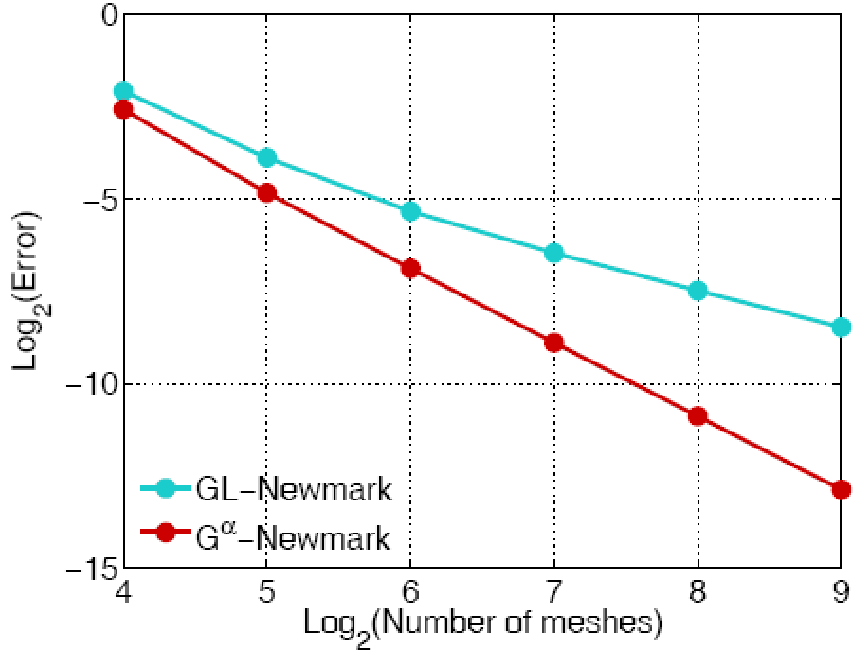
Then we use the *average acceleration algorithm:*

- **Prediction** $u_p = u^n + \Delta t\,\dot{u}^n + \frac{\Delta t^2}{4}\ddot{u}^n \quad \dot{u}_p = \dot{u}^n + \frac{\Delta t}{2}\ddot{u}^n$
- **Evaluation** $\ddot{u}^{n+1} = \frac{1}{s}\left[f^{n+1} + \phi^n - (k+\kappa)u_p^{n+1}\right]$
- **Correction** $u^{n+1} = u_p + \frac{\Delta t^2}{4}\ddot{u}^{n+1} \quad \dot{u}^{n+1} = \dot{u}_p + \frac{\Delta t}{2}\ddot{u}^{n+1}$
- **Calculation of ϕ^{n+1}**

Results

$\alpha = 1/3,\ c = 1$





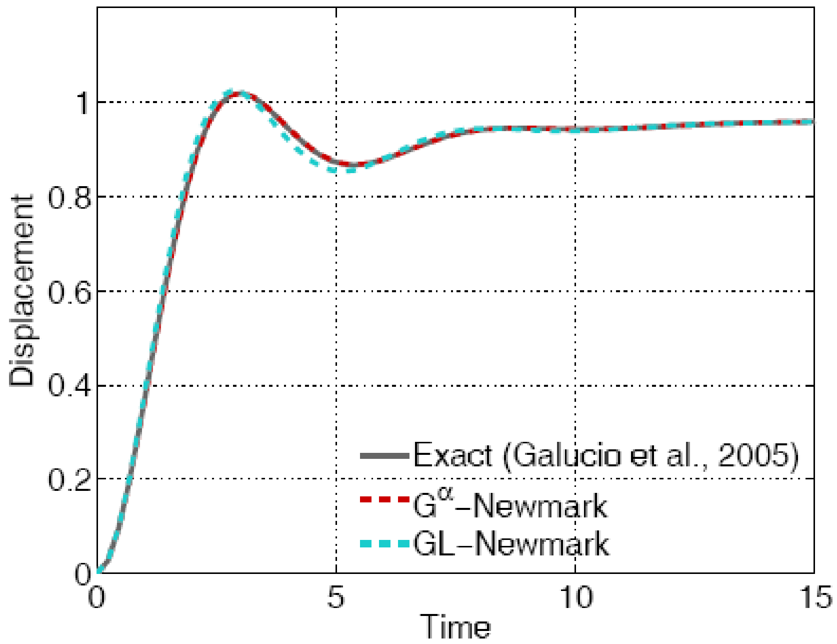
The rate of convergence computed with the L^{∞} norm is:

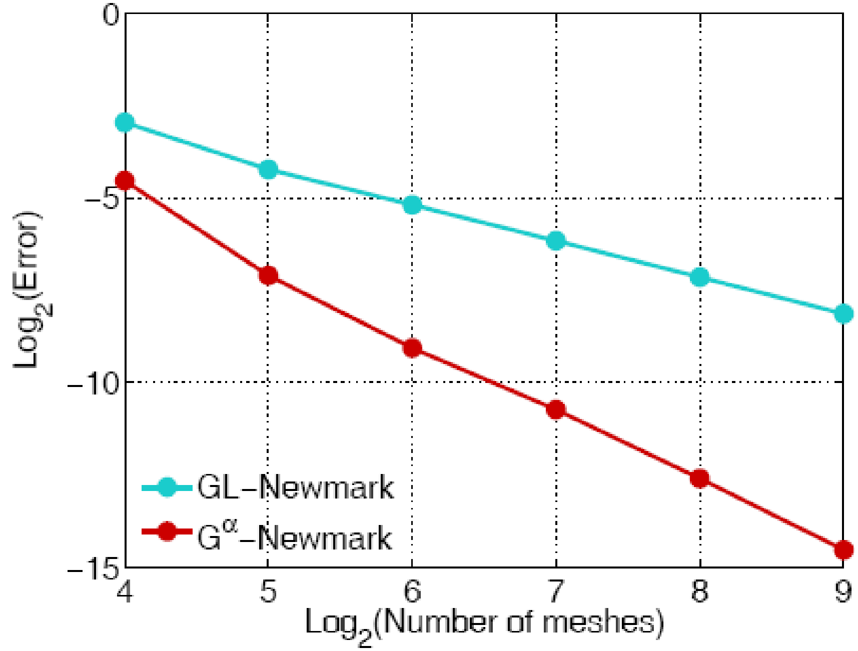
1.00 when using the GL-scheme

1.99 when using the G$^{\alpha}$-scheme

Results

$\alpha = 3/4,\ c = 1$





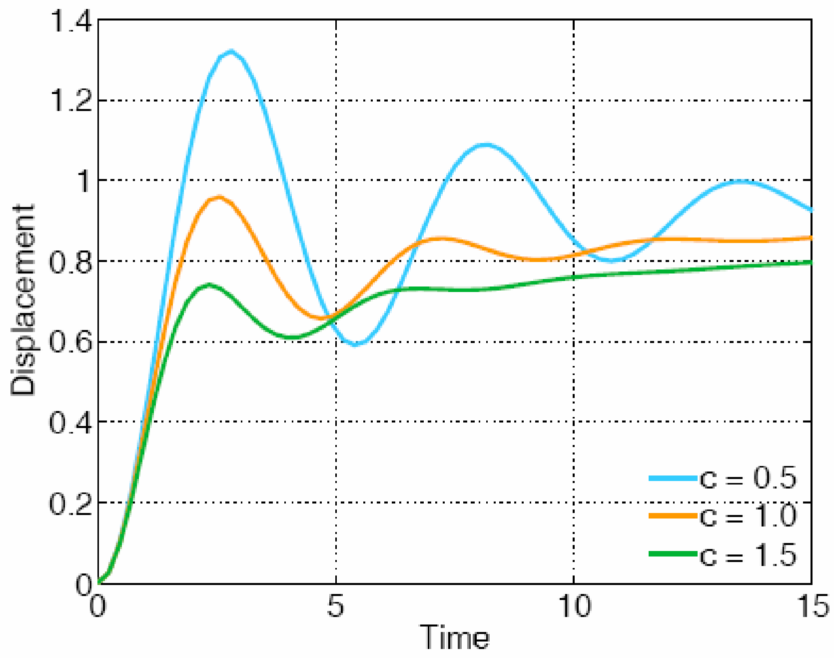
The rate of convergence computed with the L^∞ norm is:

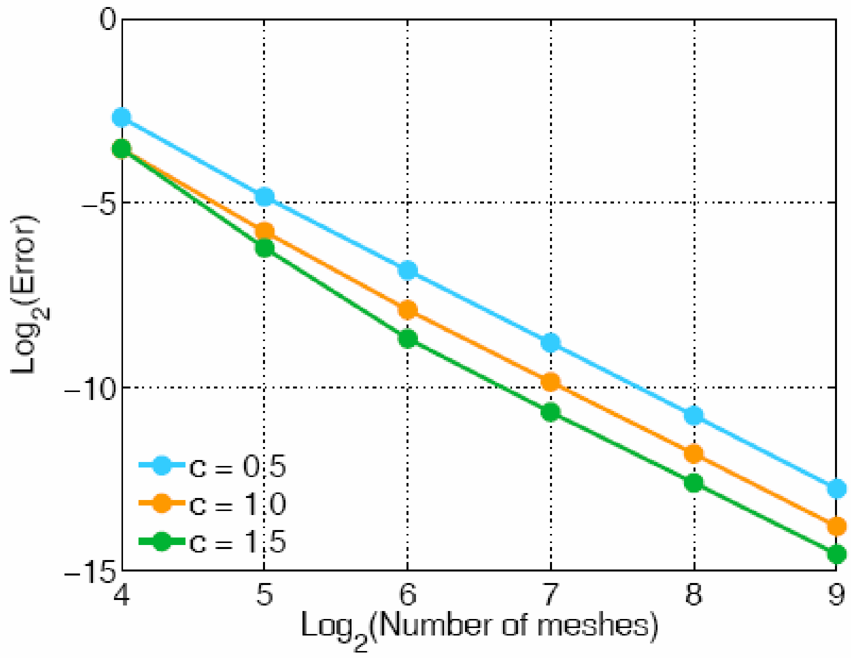
0.99 when using the GL-scheme

1.90 when using the G^α-scheme

Results

$\alpha = 1/2,\ c = 1,\ 0.5,\ 1.5$



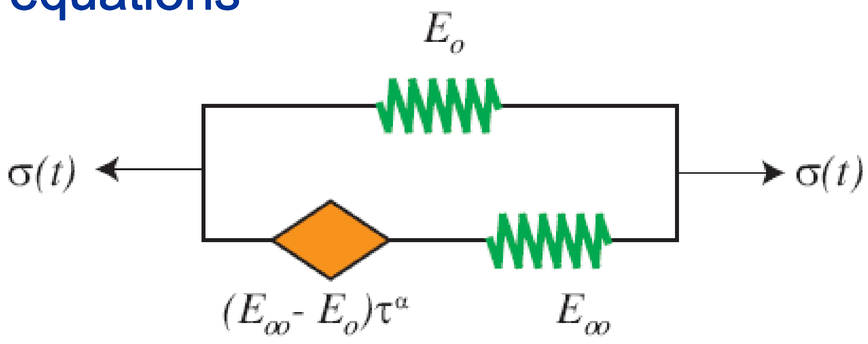


The rate of convergence is almost constant:

1.98 for $c = 0.5$ **1.96** for $c = 1.0$ **1.93** for $c = 1.5$

Extension to viscoelastic beams

Constitutive equations



Bagley & Torvik, 1984

The 1D fractional Zener model is given by:

$$\sigma(t) + \tau^\alpha \mathcal{D}^\alpha \sigma(t) = E_o \varepsilon(t) + E_\infty \tau^\alpha \mathcal{D}^\alpha \varepsilon(t)$$

Introducing the strain-type internal variable $\varepsilon_\alpha = \varepsilon - \dfrac{\sigma}{E_\infty}$,

We obtain only one fractional derivative operator:

$$\varepsilon_\alpha(t) + \tau^\alpha \mathcal{D}^\alpha \varepsilon_\alpha(t) = \frac{E_\infty - E_o}{E_\infty} \varepsilon(t)$$

Implementation in a finite element code and resolution of the dynamic problem

Viscoelastic cantilever beam

We search to solve the problem

$$\begin{cases} \mathbf{M}\ddot{\mathbf{q}}^{n+1} + (\mathbf{K} + \mathcal{K})\mathbf{q}^{n+1} = \mathbf{F}^{n+1} + \mathbf{\Phi}^{n+1} \\ \mathbf{q}^0 = \dot{\mathbf{q}}^0 = \mathbf{0} \end{cases}$$

where $\mathcal{K} = c_\alpha \dfrac{E_\infty - E_o}{E_o}\mathbf{K}$ and $\mathbf{\Phi}^{n+1} = -c_\alpha \dfrac{E_\infty}{E_o}\mathbf{K}\displaystyle\sum_{k=1}^{N} g_{k+1}\mathbf{q}_\alpha^{n+1-k}$

with $c_\alpha = \dfrac{\tau^\alpha}{\tau^\alpha + \Delta t^\alpha}$

Algorithm

We search u^{n+1} such that

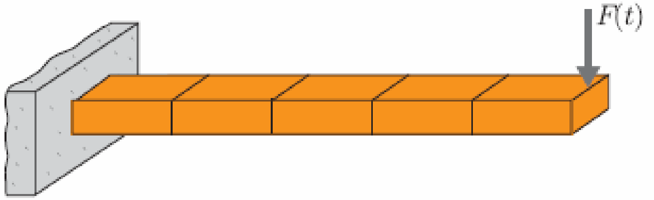
$$\begin{cases} \mathbf{M}\ddot{\mathbf{q}}^{n+1} + (\mathbf{K} + \mathcal{K})\mathbf{q}^{n+1} = \mathbf{F}^{n+1} + \mathbf{\Phi}^{n+1} \\ \mathbf{q}^0 = \dot{\mathbf{q}}^0 = \mathbf{0} \end{cases}$$

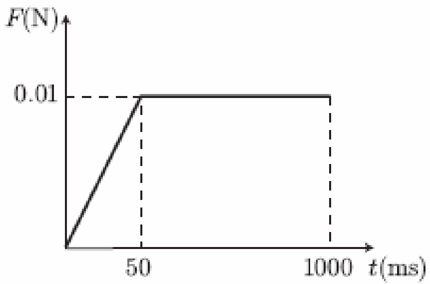
Then we use the *average acceleration algorithm*, which requires the calculation:

- $\mathbf{\Phi}^{n+1} = -c_\alpha \dfrac{E_\infty}{E_o} \mathbf{K} \displaystyle\sum_{k=1}^{N} g_{k+1} \mathbf{q}_\alpha^{n+1-k}$
- $\mathbf{q}_\alpha^{n+1} = (1 - c_\alpha) \dfrac{E_\infty - E_o}{E_\infty} \mathbf{q}^{n+1} - c_\alpha \displaystyle\sum_{k=1}^{N} g_{k+1} \mathbf{q}_\alpha^{n+1-k}$

Results

Finite element implementation of a viscoelastic beam submitted to a mechanical load





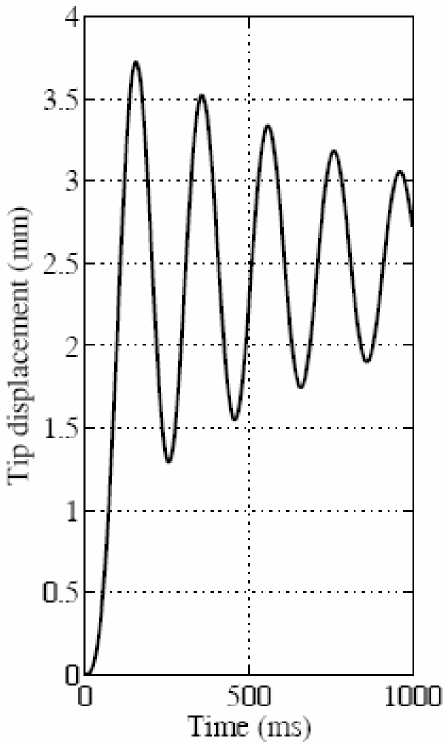
$\alpha = 0.5$

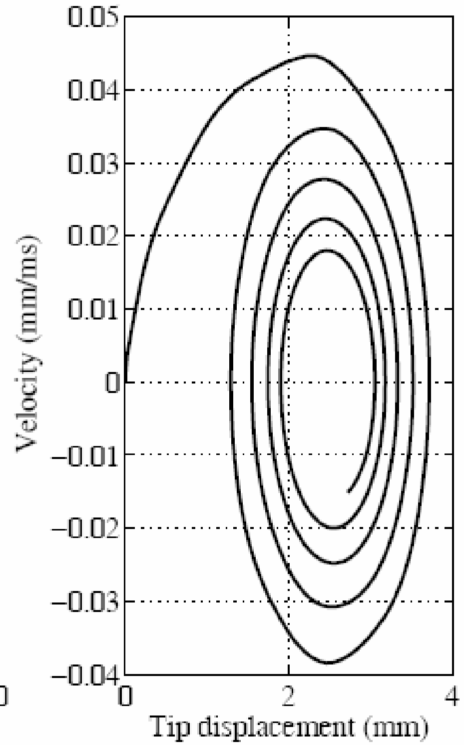
$E_o = 1$ MPa

$E_\infty = 50$ MPa

$\tau = 1$ ms

$\Delta t = 2$ ms





The whole history of the *anelastic* displacements is used in the calculations.

Quelques références

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Conclusions and Perspectives

✓ Sandwich beam with viscoelastic core and elastic faces

✓ FE implementation of a viscoelastic fractional derivative model

✓ The G^{α} scheme is no more costly than the Grünwald-Letnikov one and it appears to be $(\alpha+2)$-order accurate for power functions

✓ The combination Newmark- G^{α} yields a two-order accuracy for the damped oscillator problem

Coming work: to investigate the order of convergence obtained by using the G^{α} scheme in the case of sandwich beams