Fractional and irrational differential systems: approximation and optimization

T. HÉLIE

Collaboration with D.Matignon and R. Mignot

— Fractional Derivatives for Mechanical Engineering - State-of-the-art and Applications —

1. **Introduction : zoology and basic ideas**
2. **Systems under consideration**
   - Integral representations with poles and cuts
   - Finite-dimensional approximation by interpolation
3. **Specialized optimization procedures**
   - Functional spaces and measures
   - Regularized criterion with equality constraints
   - Numerical optimization
4. **Applications**
   - Fractional systems
   - Irrational systems
5. **Conclusion and Perspectives**

Outline

Zoology of $\xrightarrow{e(t)}$ Fractional(/Irrational) Syst. $\xrightarrow{y(t)}$

Fractional/Irrational syst.	**Transfer fct.** (analytic in $\Re e(s) > 0$)
Integrator $I_{1/2}$	$H_1(s) = 1/\sqrt{s}$ $(\rightarrow H(s)^2 = 1/s)$
Derivative $\partial_t^{1/2}$	$H_2(s) = \sqrt{s}$ $(\rightarrow H(s)^2 = s)$
Frac. Diff. Eq. $(0 < \alpha < 1)$ $\sum_{p=0}^{P} \partial_t^{p\alpha} y = \sum_{q=0}^{Q} \partial_t^{q\alpha} e$	$H_3(s) = \sum_{q=0}^{Q} s^{q\alpha} / \sum_{p=0}^{P} s^{p\alpha}$

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Bessel : $y(t) = [J_0 \star u](t)$	$H_4(s) = 1/\sqrt{s^2+1}$
Fract. PDE : $(\partial_z + \partial_t^{1/2})x = 0$ $y(t) = x(z,t), \ \partial_z x(0,t) = -e(t)$	$H_5(s) = e^{-\sqrt{s}z}/\sqrt{s}$
Flared lossy acoustic pipe	$H_6(s) = 2\Gamma(s)e^{s-\Gamma(s)}/[s+\Gamma(s)]$ with $\Gamma(s) = \sqrt{s^2+\varepsilon s^{3/2}+1}$

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$\rightarrow$ **singularities of $H_k(s)$:** poles and **cuts** in $\Re e(s)<0$

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  - **(i)** Causal stable system $\Rightarrow$ H analytic in $\Re e(s) > 0$
  - **(ii)** It is “natural” to preserve the Hermitian symmetry since a real system $\Rightarrow H_1(\bar{s}) = \overline{H_1(s)}$ in $\Re e(s) > 0$

Basic idea : adapted Bromwich contour

Let $e_+^t = e^t \mathbf{1}_{\mathbb{R}^+}(t)$ be the causal exponential.

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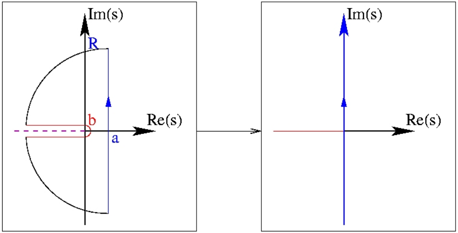

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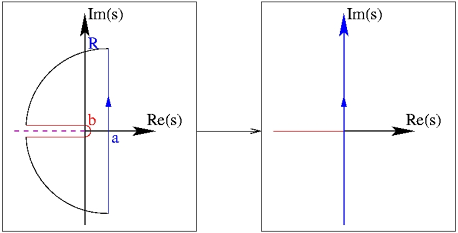

- $\boxed{h(t) + 0 - \int_0^{+\infty} \mu(-\xi)\mathrm{e}^{-\xi t}_+ \mathrm{d}\xi + 0 = 0}$ with

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- Transfer function : aggregation of first order systems

$$F(-\xi,s) = \frac{\Phi(-\xi,s)}{E(s)} = \frac{1}{s+\xi}, \quad \forall\xi > 0$$

$$H_1(s) = \frac{Y(s)}{E(s)} = \frac{\int_0^{+\infty}\mu(-\xi)\Phi(-\xi,s)\mathrm{d}\xi}{E(s)} = \int_0^{+\infty}\mu(-\xi)F(-\xi,s)\mathrm{d}\xi$$

$$= \int_0^{+\infty} \frac{\mu(-\xi)}{s+\xi}\mathrm{d}\xi \left(= \frac{1}{\sqrt{s}}\right), \quad \text{for } \Re e(s) > 0$$

Basic idea : generalizations and questions

Summary :

- Determine the singularities (poles and cuts) of H(s).

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Questions :

- Are such integral representations always well-posed ?
- How to perform accurate approximations and simulations in the time domain ?

Outline

Definitions

- Many transfer functions can be decomposed as follows, in some right-half complex plane $\mathbb{C}_a^+ := \{\Re e(s) > a\}$,

$$H(s) = \sum_{k=1}^{K} \sum_{l=1}^{L_k} \frac{r_{k,l}}{(s - s_k)^l} + \int_{\mathcal{C}} \frac{M(\mathrm{d}\gamma)}{s - \gamma} \, ,$$

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- The integral part can be realized by a dynamical system :

$$\begin{aligned} \partial_t \phi(\gamma, t) &= \gamma \, \phi(\gamma, t) + u(t), \quad \phi(\gamma, 0) = 0, \qquad \forall \gamma \in \mathcal{C} \\ y(t) &= \int_{\mathcal{C}} \phi(\gamma, t) \, M(\mathrm{d}\gamma), \end{aligned}$$

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- A well-posedness condition must be fulfilled :

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- Note the hermitian symmetry property :

$$H(s) = \overline{H(\overline{s})}, \forall s \in \mathbb{C}_a^+$$

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- Approximation of the state $\phi(\gamma, t)$, for $\{\gamma_p\}_{0\leq p\leq P+1} \subset \mathcal{C}$
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- Convergence results can be proved, as dim. $P \longrightarrow \infty$.

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- Norms in L^2, or Sobolev spaces H^s, are defined as :

$$\|h\|^2_{H^s(\mathbb{R}_t)} = \int_{\mathbb{R}_f} w_s(f)\, |H(2i\pi f)|^2 \,\mathrm{d}f\ , \text{ with } w_s(f) = (1+4\pi^2 f^2)^s\ .$$

where $s \in \mathbb{R}$ tunes the balance between low and high frequencies.

Re-interpreting Sobolev spaces

- Optimization in the frequency domain, stemming from

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  where $s \in \mathbb{R}$ tunes the balance between low and high frequencies.

- For specific applications, more general frequency dependent weights can be used : bounded frequency range, logarithmic scale, relative error measurement, bounded dynamics ...

Building up specific weights for audio applications

For audio applications, $w(f)$ can be adapted and modified according to the following requirements :

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4. a relative error on a bounded dynamics : $w(f)/(\mathrm{Sat}_{H,\Theta}(f))^2$ where the saturation function $\mathrm{Sat}_{H,\Theta}$ with threshold Θ is defined by

$$\mathrm{Sat}_{H,\Theta}(f) = \begin{cases} |H(2i\pi f)| & \text{if } |H(2i\pi f)| \geq \Theta_H \\ \Theta_H & \text{otherwise} \end{cases}$$

Note : normalization of the samples is desirable in most audio applications, before the sequence is sent to DAC audio converters.

Regularized criterion with equality constraints

- The regularized criterion reads :

$$\mathcal{C}_R(\mu) = \int_{\mathbb{R}^+} \left| \widetilde{H_\mu}(2i\pi f) - H(2i\pi f) \right|^2 w(f)\mathrm{d}f + \sum_{p=1}^{P} \epsilon_p |\mu_p|^2,$$

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- Equality constraints for $\widetilde{H_\mu}^{(d_j)}$ at prescribed frequency points η_j, $1 \leq j \leq J$ are taken into account thanks to a Lagrangian $\mathcal{C}_{R,L}$ by adding to $\mathcal{C}_R$:

$$\Re\mathrm{e} \left(\ell^* \begin{bmatrix} H^{(d_1)}(2i\pi\eta_1) - \widetilde{H_\mu}^{(d_1)}(2i\pi\eta_1) \\ \vdots \\ H^{(d_J)}(2i\pi\eta_J) - \widetilde{H_\mu}^{(d_J)}(2i\pi\eta_J) \end{bmatrix} \right),$$

Discrete criterion

- Discrete version of the criterion for frequencies increasing from $f_1 = f_- $ to $f_{N+1} = f_+$ is, with $s_n = 2i\pi f_n$:

$$\mathcal{C}(\mu) \approx \sum_{n=1}^{N} w_n \left| \widetilde{H_\mu}(s_n) - H(s_n) \right|^2 \text{ with } w_n = \int_{f_n}^{f_{n+1}} w(f) \mathrm{d}f .$$

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- In matrix notations, this rewrites

$$\mathcal{C}_{R,L}(\boldsymbol{\mu}) = (\boldsymbol{M\mu} - \boldsymbol{h})^* \boldsymbol{W} (\boldsymbol{M\mu} - \boldsymbol{h}) + \boldsymbol{\mu}^t \boldsymbol{E} \boldsymbol{\mu} + \Re\mathrm{e}\left(\ell^* \left[\boldsymbol{k} - \boldsymbol{N\mu} \right] \right),$$

with

$\boldsymbol{M}$:	model	$N \times (P + P_2)$
$\boldsymbol{N}$:	constraint model	$J \times (P + P_2)$
$\boldsymbol{E}$:	regularization	$(P + P_2) \times (P + P_2)$
$\boldsymbol{W}$:	weights	$N \times N$
$\boldsymbol{h}$:	data	$N \times 1$
$\boldsymbol{k}$:	constaints	$J \times 1$

Closed-form solution

- If $J = 0$ (no constraint), the solution reduces to

$$\boldsymbol{\mu} = \mathcal{M}^{-1}\mathcal{H},$$

where $\mathcal{M} = \Re\mathrm{e}\left(\boldsymbol{M}^{*}\boldsymbol{W}\boldsymbol{M} + \boldsymbol{E}\right)$ and $\mathcal{H} = \Re\mathrm{e}\left(\boldsymbol{M}^{*}\boldsymbol{W}\boldsymbol{h}\right)$.

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- For $J \geq 1$, the solution reads :

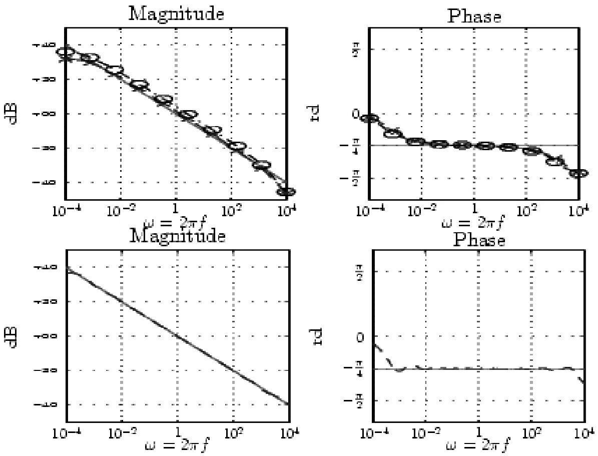
$$\boldsymbol{\mu} = \mathcal{M}^{-1}\left[\mathcal{H} + \underline{\boldsymbol{N}}^{t}\mathcal{N}^{-1}\left(\underline{\boldsymbol{k}} - \underline{\boldsymbol{N}}\mathcal{M}^{-1}\mathcal{H}\right)\right],$$

where $\mathcal{N} = \underline{\boldsymbol{N}}\mathcal{M}^{-1}\underline{\boldsymbol{N}}^{t}$ is invertible for non-redundant constraints, and $\left\{\begin{array}{lll} \underline{\boldsymbol{N}}^{t} & \text{denotes} & [\Re\mathrm{e}(\boldsymbol{N}^{t}), \Im\mathrm{m}(\boldsymbol{N}^{t})] \\ \underline{\boldsymbol{k}}^{t} & \text{denotes} & [\Re\mathrm{e}(\boldsymbol{k}^{t}), \Im\mathrm{m}(\boldsymbol{k}^{t})] \end{array}\right.$.

Outline

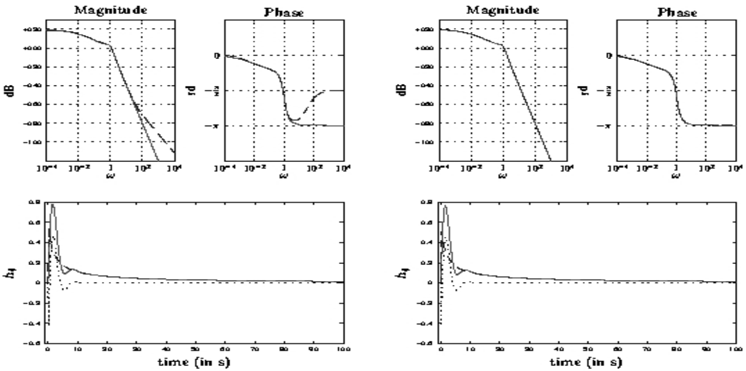
Fractional systems

An academic example : $H_1(s) = 1/\sqrt{s},\ \mu_1(-\xi) = 1/(\pi\sqrt{\xi})$



Top : Interpolation, $P = 16$. Bottom : Optimization, $P = 10$!

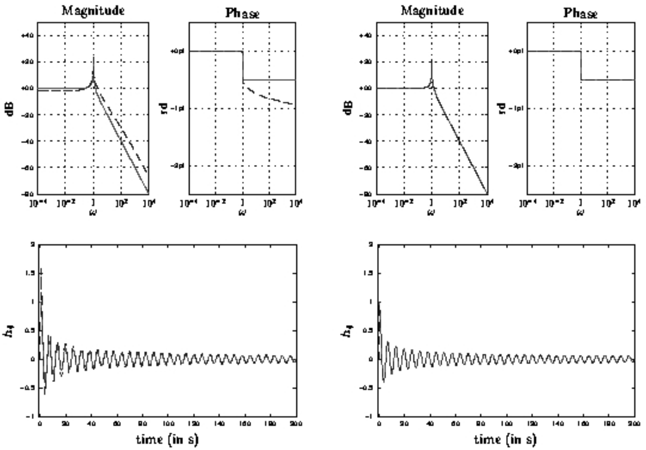
Fractional AR : $H_3(s) = 1/(s^2 + 0.1s^{3/2} + s^{1/2} + 0.1)$ (poles and $\mathbb{R}^-$)



Left : Interpolation, $P = 18$. Right : Optimization, $P = 18$!
$(\ldots)$: poles only. $(- -)$: cut only. $(-)$: poles and cut.

Bessel kernel : 2 cuts $\pm i + \mathbb{R}^-$

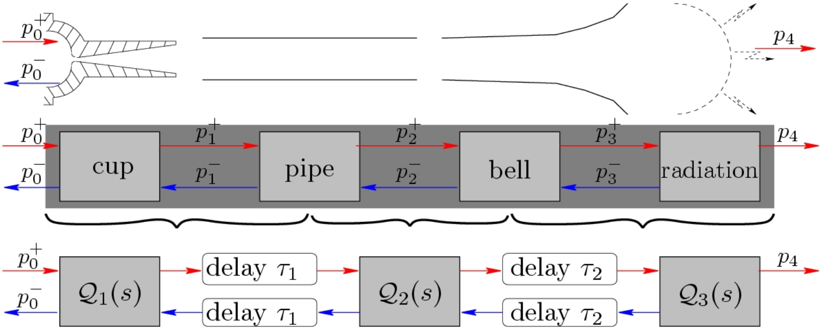
$$H_4(s) = 1/\sqrt{s^2+1}, \quad \mu_4^{\pm}(-\xi) = 1\,/\,(\pi\sqrt{\xi(\pm 2i - \xi)})$$



Left : Interpolation, $P = 10$. Right : Optimization, $P = 10$!

Trumpet-like instrument (I)

Decomposition into elementary subsystems.

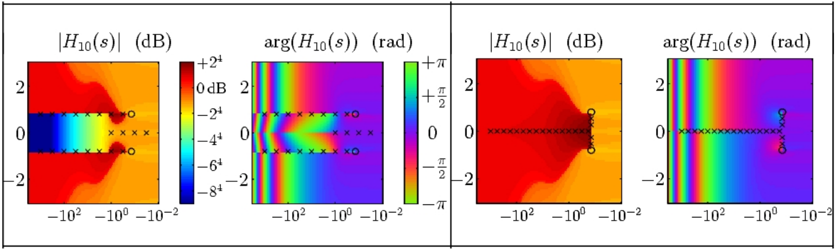


Transfer functions of interest :

- Reflection between p_0^+ and p_0^-.
- Transmission between p_0^+ and p_4.

Trumpet-like instrument (II) : various choices of the cuts

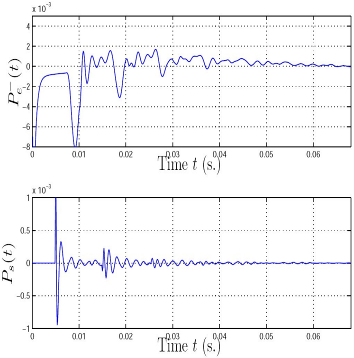
- with 3 Horizontal cuts, with a Cross cut



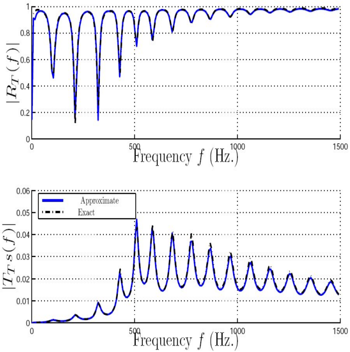
- Remark : the values of $H(s)$ in $\mathbb{C}_0^+$ do not depend on the choice of the cut !

Trumpet-like instrument (III)

Time-domain representation



Frequency-domain rep.



Real-time simulations in Pure-Data environment on optimized models with $P \leq 10$ for each quadripole $\mathcal{Q}_k$: bounded freq. range, log-scale & relat. error.

Outline

Perspectives

- Open question : choice of the cut ?

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- Open question : choice of the cut ?
- Open question : optimal placement of the poles, once the cut has been chosen ?
- What can *not* be represented by poles and cuts ?
  - *Delay* systems stemming from wave *propagation* phenomena.
  - systems of PDEs with *variable* coefficients : must be decomposed into subsystems with constant coefficients.

Conclusion

- A powerful and very flexible method of simulation of some *infinite*-dimensional linear systems has been presented : it uses a simple optimization procedure with parameters which are meaningful from a signal processing point of view, and it enables a low cost simulation (both in the frequency domain and in the time domain), even suitable for real-time applications.

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- From a a theoretical point of view, this method is based on a *representation with poles and cuts*, which generalizes the so-called diffusive representations.
- Many such systems, among which fractional differential systems, have been presented here and elsewhere, which clearly illustrates the generality, the flexibility and the power of this method.

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