Fractional derivatives: Basic definitions

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Outline

Family of causal fractional powers

Definition

$$\text{For} \quad \alpha > 0, \quad Y_\alpha(t) \triangleq \frac{1}{\Gamma(\alpha)} t_+^{\alpha-1} \in L^1_{\text{loc}}(\mathbb{R}^+) \tag{1}$$

where Γ is the Euler gamma function.
Y_n is a polynomial of degree $n-1$ and Y_1 is Heaviside step fct.

Proposition

We have the important convolution property:

$$\forall \alpha > 0, \forall \beta > 0, \quad Y_\alpha \star Y_\beta = Y_{\alpha+\beta} \tag{2}$$

Proof: by setting $t = x\,\tau$.

$$\int_0^t (t-\tau)^{\alpha-1} \tau^{\beta-1}\, d\tau = t^{\alpha+\beta-1} \int_0^1 (1-x)^{\alpha-1} x^{\beta-1}\, dx$$

■

Fractional integral

Definition

The fractional integral of order $\alpha > 0$ of a *causal* function f is:

$$I^{\alpha} f \triangleq Y_{\alpha} \star f \quad (3)$$

$$I^{\alpha} f(t) = \int_0^t \frac{1}{\Gamma(\alpha)\,(t-\tau)^{1-\alpha}}\, f(t-\tau)\, d\tau \quad (4)$$

Proposition

Convolution property (2) translates into a sequentiality property:

$$\forall \alpha > 0,\ \forall \beta > 0, \quad I^{\alpha} \circ I^{\beta} = I^{\alpha+\beta} \quad (5)$$

Smooth inversion ?

Definition

The Caputo fractional derivative of order $0 < \alpha < 1$ of a *causal* function g is:

$$d^\alpha g = D_C^\alpha g \triangleq Y_{1-\alpha} \star \frac{dg}{dt} \tag{6}$$

$$d^\alpha g(t) = \int_0^t \frac{1}{\Gamma(1-\alpha)\,(t-\tau)^\alpha} \frac{dg}{dt}(\tau)\,d\tau \tag{7}$$

Proposition

Caputo takes into account initial values, since:

$$d^\alpha \circ I^\alpha f = f - f(0^+) \tag{8}$$

It proves appropriate for problems with initial conditions.

How the proof goes (1)

Proof.

Since $g = Y_\alpha \star f$, we have $g' = Y_\alpha \star f'$.
Then $Y_{1-\alpha} \star g' = Y_{1-\alpha} \star (Y_\alpha \star f') = (Y_{1-\alpha} \star Y_\alpha) \star f'$.
But (2) $\Rightarrow Y_{1-\alpha} \star Y_\alpha = Y_1$, and $\int_0^t f'(\tau)\,d\tau = f(t) - f(0^+)$. ∎

Remark

Following $g'' = (g')'$ in the integer case, in the sequel, we will make use of a composition rule for Caputo derivatives, namely $(d^\alpha)^{\circ 2} = d^\alpha \circ d^\alpha$ and so on, in order to take into account as many fractional initial conditions as needed.
But, there is no need that $(d^\alpha)^{\circ 2} g$ and $d^{2\alpha} g$ coincide!

Less smooth inversion !

Definition

The Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ of a *causal* function g is:

$$D_{RL}^{\alpha} g \triangleq \frac{d}{dt} \left(Y_{1-\alpha} \star g \right) \tag{9}$$

$$D_{RL}^{\alpha} g(t) = \frac{d}{dt} \int_0^t \frac{1}{\Gamma(1-\alpha)\,(t-\tau)^{\alpha}}\, g(\tau)\, d\tau \tag{10}$$

Proposition

Riemann-Liouville derivative is a right inverse to fractional integral:

$$D_{RL}^{\alpha} \circ I^{\alpha} f = f \tag{11}$$

How the proof goes (2)

Proof.

Now $Y_{1-\alpha} \star g = Y_{1-\alpha} \star (Y_\alpha \star f) = (Y_{1-\alpha} \star Y_\alpha) \star f = Y_1 \star f$.
Then $\frac{d}{dt} \int_0^t f(\tau)\, d\tau = f(t)$. ■

Proposition (Link between C and RL)

$$D_{RL}^\alpha g = d^\alpha g + g(0^+)\, Y_{1-\alpha}(t), \tag{12}$$

Riemann-Liouville takes into account intial conditions but in a non smooth way, since $Y_{1-\alpha}(t) \propto t^{-\alpha} \to +\infty$ as $t \to 0^+$. Thus, it is most unappropriate for problems with initial conditions!

Within the framework of causal distributions (1)

Remark

Note that, like $d^1 Y_1 = 0$, $D^\alpha_{RL} Y_\alpha = 0$. But $I^\alpha 0 = 0$, thus I^α and D^α_{RL} cannot be inverses one of another!
$\Rightarrow$ indeed, for consistency reasons, Dirac pulses must fit into the framework.

Definition

The Schwartz fractional derivative of order $0 < \alpha < 1$ of a *causal* function or distribution g is:

$$D^\alpha g \triangleq Y_{-\alpha} \star g \quad \text{with } Y_{-\alpha} \triangleq \text{fp}\left\{\frac{1}{\Gamma(-\alpha)}\, t_+^{-1-\alpha}\right\}, \tag{13}$$

where fp means finite part in the sense of Hadamard.
Note the special case $Y_{-n} = \delta^{(n)}$, for $n \in \mathbb{N}$.

Within the framework of causal distributions (2)

Proposition

Let $\mathcal{D}'_+$ the space of causal distributions, i.e. with support in $\mathbb{R}^+$.

- *$\mathcal{D}'_+$ is a convolution algebra,*
- *$Y_\alpha \star Y_\beta = Y_{\alpha+\beta}$, for all $\alpha, \beta \in \mathbb{R}$,*
- *$Y_{-\alpha} \star Y_\alpha = Y_0 = \delta$, the unit element of convolution,*
- *D^α is the inverse of I^α.*

Let g a causal and regular function, $D^1 g = g' + g(0^+)\,\delta$, and since $D^\alpha = I^{1-\alpha} \circ D^1$ we get:

$$D^\alpha g = Y_{1-\alpha} \star g' + g(0^+)\,Y_{1-\alpha}.$$

Does $D^\alpha g = D^\alpha_{RL} g$?

An example for $\alpha = 1/2$ (1)

Let $g = b_0 + b_1\sqrt{t} + b_2 t + b_3 t^{3/2} + b_4 t^2$ a causal function. With the help of the $Y_{k/2}$ family, this expansion reads:

$$
\begin{aligned}
g &= a_0 Y_1 + a_1 Y_{3/2} + a_2 Y_2 + a_3 Y_{5/2} + a_4 Y_3 . \\
D^{1/2} g &= a_0 Y_{1/2} + \underbrace{a_1 Y_1 + a_2 Y_{3/2} + a_3 Y_2 + a_4 Y_{5/2}}_{d^{1/2} f} \\
D^1 g &= a_0 Y_0 + \underbrace{a_1 Y_{1/2} + \underbrace{a_2 Y_1 + a_3 Y_{3/2} + a_4 Y_2}_{(d^{1/2})^2 f}}_{d^1 f} \\
D^{3/2} g &= a_0 Y_{-1/2} + a_1 Y_0 + a_2 Y_{1/2} + \underbrace{a_3 Y_1 + a_4 Y_{3/2}}_{(d^{1/2})^3 f} \\
D^2 g &= a_0 Y_{-1} + a_1 Y_{-1/2} + a_2 Y_0 + a_3 Y_{1/2} + \underbrace{a_4 Y_1}_{(d^{1/2})^4 f}
\end{aligned}
$$

An example for $\alpha = 1/2$ (2)

Proposition

Since $(d^{1/2})^{\circ k} g$ are continuous at $t = 0^+$, it is easily seen that

$$a_k = [(d^{1/2})^{\circ k} g](t = 0^+).$$

And the following fractional and causal Taylor expansion holds:

$$g = \sum_{k=0}^{4} a_k Y_{1+k/2}.$$

Remark

Also problems pop up with Riemann-Liouville definition:

- $D^1_{RL} g \neq D^1_{RL} g$ because a_0 drops out
- $D^{1/2}_{RL} D^1 g \neq D^{3/2} g$ because a_1 drops out
- With RL, all terms in $Y_{-n} = \delta^{(n)}$, $n \geq 0$ drop out

Outline

Definitions

Proposition

For $\alpha > 0$, the Laplace transform of Y_α is:

$$\mathcal{L}[Y_\alpha](s) = s^{-\alpha} \quad \textit{for} \quad \Re e(s) > 0 \tag{14}$$

Proof: first for real s, setting $x = ts$.

$$\mathcal{L}[Y_\alpha](s) := \int_0^{+\infty} \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-st}\, dt = s^{-\alpha} \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} x^{\alpha-1} e^{-x}\, dx = s^{-\alpha}$$

from the definition of Γ. In the complex plane, the result can be extented to the strip $\Re e(s) > 0$ by analyticity. ■

Fractional powers of complex numbers?

Remark

1. In (14), the function $s \mapsto s^\alpha$ of the complex variable s is defined with no ambiguity as the analytic continuation of $x \mapsto x^\alpha$ on $\mathbb{R}^+$ into the *strip* of convergence of the Laplace transform, namely $\Re e(s) > 0$.
2. Hence, for $s = \rho \exp(i\theta)$ and $|\theta| < \pi/2$, s^α has the analytic value $s^\alpha = \rho^\alpha \exp(i\alpha\theta)$.
3. The value on the imaginary axis is uniquely determined.
4. In the left-half plane, the values are to be defined. But the function is multiform: when $\theta \mapsto \theta + 2\pi$, $s^\alpha_{\theta+2\pi} \neq s^\alpha_\theta$ since $\alpha \notin \mathbb{N}$. A cut must be drawn in $\Re e(s) < 0$ for the function to be uniform: $\mathbb{R}^-$ is a convenient choice.

Fractional integrals as filters

Definition

For $0 < \alpha < 1$, the Fourier transform of Y_α is:

$$\mathcal{F}[Y_\alpha](f) = (2i\pi f)^{-\alpha} = |2\pi f|^{-\alpha} \exp(-i\alpha\frac{\pi}{2}\,\mathrm{sign}(f)) \quad (15)$$

- the magnitude behaves like -6α dB/oct,
- the phase is locked to $\mp\alpha\frac{\pi}{2}$ rad.

Corollary

Rewriting $y = I^\alpha u = Y_\alpha \star u$ in the Fourier domain reads $\hat{y}(f) = \mathcal{F}[Y_\alpha]\,.\,\hat{u}(f)$.
Thus, fractional integral is nothing but a low-pass filter of non-integer order α.

Fractional derivatives as filters

Definition

For $0 < \alpha < 1$, the Fourier transform of $Y_{-\alpha}$ is:

$$\mathcal{F}[Y_{-\alpha}](f) = (2i\pi f)^{+\alpha} = |2\pi f|^{+\alpha} \exp(+i\alpha\frac{\pi}{2}\,\mathrm{sign}(f)) \qquad (16)$$

- the magnitude behaves like $+6\alpha$ dB/oct,
- the phase is locked to $\pm\alpha\frac{\pi}{2}$ rad.

Corollary

Rewriting $y = D^\alpha u = Y_{-\alpha} \star u$ in the Fourier domain reads $\hat{y}(f) = \mathcal{F}[Y_{-\alpha}]\,.\,\hat{u}(f)$.
Thus, fractional derivative is nothing but a high-pass filter of non-integer order α.

Outline

The scalar case

Definition

For $0 < \alpha \leq 1$ and $\lambda \in \mathbb{C}$, $E_\alpha(\lambda, t)$ is the eigenfunction of d^α for the eigenvalue λ, initialized to 1.

$$\begin{cases} d^\alpha E_\alpha(\lambda, t) &= \lambda E_\alpha(\lambda, t), \quad t > 0 \\ E_\alpha(\lambda, 0^+) &= 1 \end{cases} \tag{17}$$

Proposition

$$E_\alpha(\lambda, t) = \sum_{k=0}^{\infty} \lambda^k \, Y_{1+\alpha k}(t) = E_\alpha(\lambda t_+^\alpha) \tag{18}$$

with $E_\alpha(z)$ the Mittag-Leffler fct. defined by a power series:

$$E_\alpha(z) := \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(1 + \alpha k)} \tag{19}$$

The vector case

Definition

For $0 < \alpha \leq 1$ and $\lambda \in \mathbb{C}$, $E_\alpha(\Lambda, t)$ is the matrix-valued eigenfunction of d^α for the eigenvalue $\Lambda \in M_n(\mathbb{C})$, initialized to the identity matrix I_n.

$$\begin{cases} d^\alpha E_\alpha(\Lambda, t) &= \Lambda E_\alpha(\Lambda, t), \quad t > 0 \\ E_\alpha(\Lambda, 0^+) &= I_n \end{cases} \tag{20}$$

Proposition

$$E_\alpha(\Lambda, t) = \sum_{k=0}^{\infty} \Lambda^k \, Y_{1+\alpha k}(t) = E_\alpha(\Lambda t_+^\alpha) \tag{21}$$

For $\alpha = 1$, this is the exponential of the square matrix Λ.

Stability result

Remark

$$E_{1/2}(\lambda, t) = \exp(\lambda^2 t)\,[1 + \operatorname{erf}(\lambda\sqrt{t})]$$

where erf is the error function, to be evaluated in the whole complex plane...

Theorem

$$(E_\alpha(\lambda, t) \to 0 \quad as \quad t \to +\infty) \quad \Longleftrightarrow \quad |\arg\lambda| > \alpha\frac{\pi}{2};$$

in which case, it has the following long-memory asymptotics:

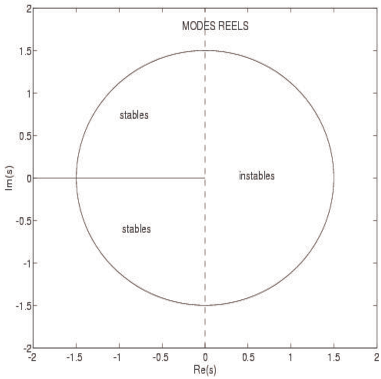
$$E_\alpha(\lambda, t) \sim K_{\lambda,\alpha}\, t^{-\alpha} \quad as \quad t \to +\infty.$$

N.B. For $\alpha = 1$, we get the stability of exponentials, but the stable behaviour is qualitatively very different: short-memory.

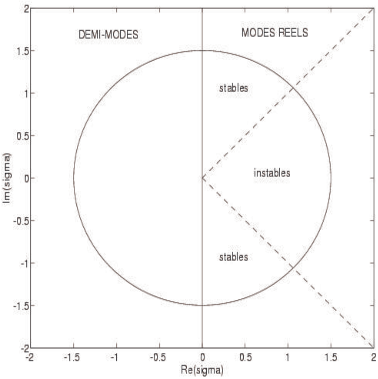
CNS de stabilité en fractionnaire commensurable

Stabilité de $E_\alpha(\lambda\, t^\alpha)$ de TL $s^{\alpha-1}(s^\alpha-\lambda)^{-1}$ en fct. de $\arg(\lambda)$.

Plan de Laplace en s



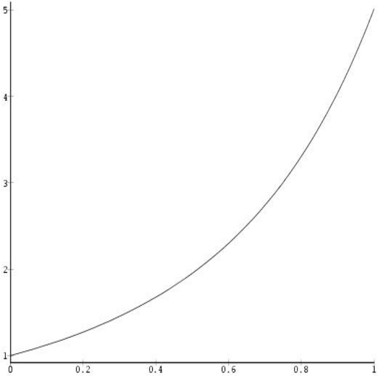
Plan en σ



Fonctions de Mittag-Leffler dans $\mathbb{C}$ (I)

$$t \mapsto E_\alpha(\lambda t^\alpha) \text{ pour } \alpha = \frac{1}{2} \text{ et } \arg(\lambda) = 0$$

Partie réelle

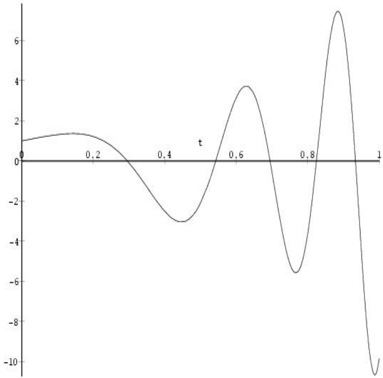


Partie imaginaire

Fonctions de Mittag-Leffler dans $\mathbb{C}$ (II)

$$t \mapsto E_\alpha(\lambda t^\alpha) \text{ pour } \alpha = \frac{1}{2} \text{ et } \arg(\lambda) = \pi/8$$

Partie réelle



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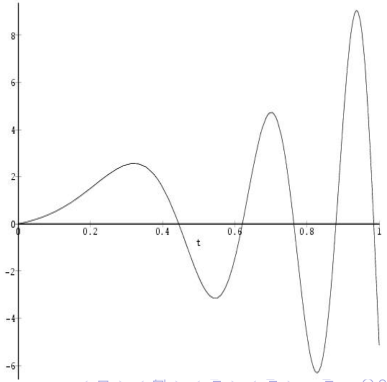
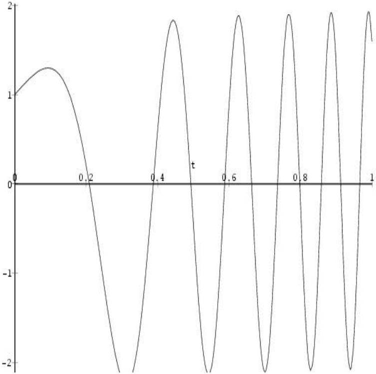


Fonctions de Mittag-Leffler dans $\mathbb{C}$ (III)

$$t \mapsto E_\alpha(\lambda t^\alpha) \text{ pour } \alpha = \tfrac{1}{2} \text{ et } \arg(\lambda) = \pi/4$$

Partie réelle



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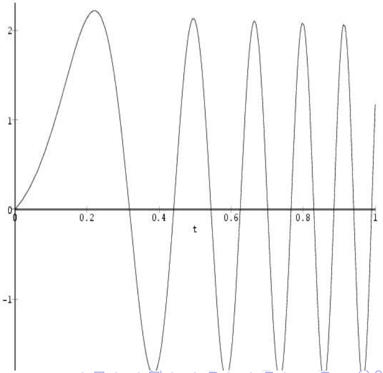
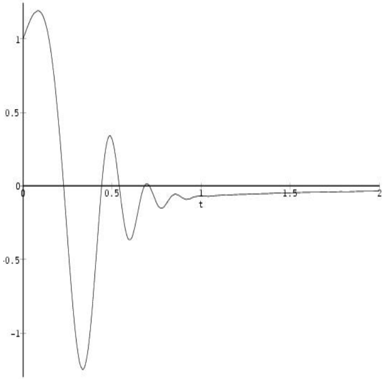


Fonctions de Mittag-Leffler dans $\mathbb{C}$ (IV)

$$t \mapsto E_\alpha(\lambda t^\alpha) \text{ pour } \alpha = \tfrac{1}{2} \text{ et } \arg(\lambda) = 3\pi/8$$

Partie réelle



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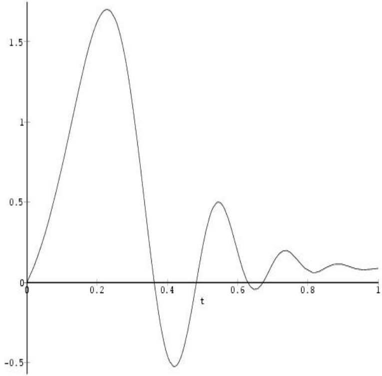
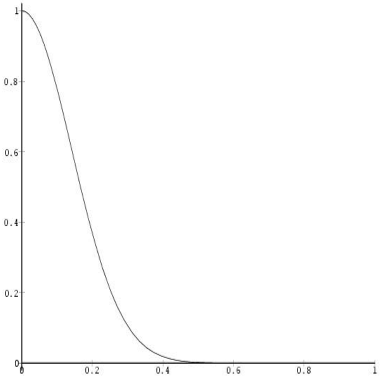


Fonctions de Mittag-Leffler dans $\mathbb{C}$ (V)

$$t \mapsto E_\alpha(\lambda t^\alpha) \text{ pour } \alpha = \tfrac{1}{2} \text{ et } \arg(\lambda) = \pi/2$$

Partie réelle



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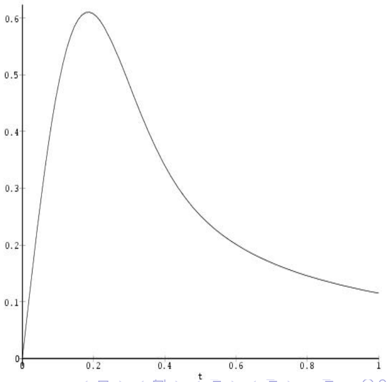
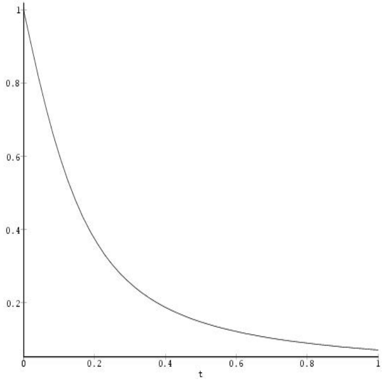


Fonctions de Mittag-Leffler dans $\mathbb{C}$ (VI)

$$t \mapsto E_\alpha(\lambda t^\alpha) \text{ pour } \alpha = \frac{1}{2} \text{ et } \arg(\lambda) = 3\pi/4$$

Partie réelle



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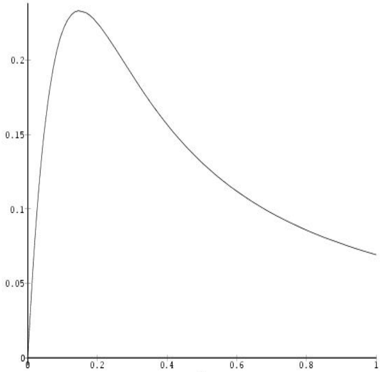
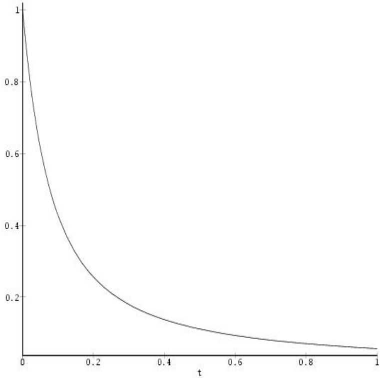


Fonctions de Mittag-Leffler dans $\mathbb{C}$ (VII)

$$t \mapsto E_\alpha(\lambda t^\alpha) \text{ pour } \alpha = \frac{1}{2} \text{ et } \arg(\lambda) = \pi$$

Partie réelle

Partie imaginaire



The scalar case (1)

Definition

For $0 < \alpha \leq 1$ and $\lambda \in \mathbb{C}$, $\mathcal{E}_\alpha(\lambda, t)$ is the fundamental solution or Green function of $D^\alpha - \lambda$ for the eigenvalue λ.

$$D^\alpha \mathcal{E}_\alpha(\lambda, t) = \lambda\, \mathcal{E}_\alpha(\lambda, t) + \delta \tag{22}$$

Proposition

$$\mathcal{E}_\alpha(\lambda, t) = \mathcal{L}^{-1}\left[(s^\alpha - \lambda)^{-1}, \Re e(s) > a_\lambda\right] = \sum_{k=0}^{\infty} \lambda^k\, Y_{(1+k)\alpha}(t) \tag{23}$$

For $\alpha = 1/2$, $\mathcal{E}_{1/2}(\lambda, t) = Y_{1/2}(t) + \lambda\, E_{1/2}(\lambda, t)$.
For $\alpha = 1$, $\mathcal{E}_1(\lambda, t) = E_1(\lambda, t) = e^{\lambda t}\, Y_1(t)$.

The scalar case (2)

Remark

Another link between the two Mittag-Leffler functions:

$$D^{\alpha} E_{\alpha}(\lambda, t) = \lambda E_{\alpha}(\lambda, t) + 1 Y_{1-\alpha}$$

Thus,

$$E_{\alpha}(\lambda, t) = Y_{1-\alpha} \star \mathcal{E}_{\alpha}(\lambda, .)$$

Definition

Also j-th convolutions of the $\mathcal{E}_{\alpha}(\lambda, .)$ functions will be used in case of multiple roots, and denoted by $\mathcal{E}_{\alpha}^{\star j}(\lambda, .)$.
N.B. When $\alpha = 1$, $(e^{\lambda t} Y_1(t))^{\star j} = Y_j(t) e^{\lambda t}$.

Stability result

Theorem (DM94)

The asymptotic behaviour of $\mathcal{E}_\alpha^{\star j}(\lambda, t)$ as $t \to +\infty$ is also given by the location λ in the σ-plane:

- *for $|\arg(\lambda)| < \alpha\frac{\pi}{2}$, $\mathcal{E}_\alpha^{\star j}(\lambda, t)$ diverges exponentially (more precisely: $\{$polynomial in $t^\alpha\} \times \{\exp(\lambda^{1/\alpha} t)\}$),*
- *for $|\arg(\lambda)| = \alpha\frac{\pi}{2}$, $\mathcal{E}_\alpha^{\star 1}(\lambda, t)$ is asymptotically oscillating; and with $j \geq 2$, $\mathcal{E}_\alpha^{\star j}(\lambda, t)$ diverges polynomially (in t^α) with oscillations,*
- *for $|\arg(\lambda)| > \alpha\frac{\pi}{2}$, $\mathcal{E}_\alpha^{\star j}(\lambda, t) \sim k_{j,\alpha}\, \lambda^{-1-j}\, t^{-1-\alpha}$ is stable.*

In this latter case, $\mathcal{E}_\alpha^{\star j}(\lambda, t) \in L^1(]0, +\infty[)$, will give rise to stable convolution systems, BIBO systems ($L^1 \star L^\infty \subset L^\infty$).

Solving a FDE (1)

Algebraïc treatment of FDEs of commensurate orders.

$$\text{Let} \quad P(\sigma) = \sigma^n + c_{n-1}\sigma^{n-1} + \cdots + c_0 = \prod_{i=1}^{r} (\sigma - \lambda_i)^{m_i}$$

Consider the following FDE with input x and unknown y:

$$(d^\alpha)^n y + c_{n-1}(d^\alpha)^{n-1} y + \cdots + c_0 y = x$$

and initial conditions $a_k = [(d^\alpha)^k y](0)$ for $0 \le k \le n-1$.

The solution reads: $y = (h \star x)(t) + \sum_{k=0}^{n-1} a_k \, e_k(t)$.

In case of distinct roots for P, the impulse response h reads:

$$h(t) := \mathcal{L}^{-1}\left[\frac{1}{P(s^\alpha)}\right] = \mathcal{E}_\alpha(\lambda_1, t) \star \cdots \star \mathcal{E}_\alpha(\lambda_n, t) = \sum_{i=1}^{n} \frac{1}{P'(\lambda_i)} \mathcal{E}_\alpha(\lambda_i, t)$$

Solving a FDE (2)

Example $n = 2$ and $\alpha = 1/2$, $P(\sigma) = \sigma^2 + c_1\sigma + c_2$.

- impulse response: $h(t) = \sum_{i=1}^{2} \frac{1}{P'(\lambda_i)} \mathcal{E}_{1/2}(\lambda_i, t)$
- response to the fractional initial condition a_1:

$$e_1(t) = I^{1/2}h = \sum_{i=1}^{2} \frac{1}{P'(\lambda_i)} E_{1/2}(\lambda_i\sqrt{t})$$

- response to the integer initial condition a_0:

$$e_0(t) = D^{1/2}e_1 + c_1\, e_1 = h + c_1\, e_1 = \sum_{i=1}^{2} \frac{\lambda_i + c_1}{P'(\lambda_i)} E_{1/2}(\lambda_i\sqrt{t})$$

Important: Solution y is $\mathcal{C}^1 \iff a_1 = 0$.
The meaning of fractional initial conditions is now understood.

Outline

Limitations du cadre commensurable

1. Caractère nécessairement commensurable des ordres de dérivation, fractionnaires ou non,

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2. Notion d'état assez fictive, très formelle, qui ne représente rien de concret physiquement, c'est une étape algébrique : la dimension (arbitrairement élevée si $\alpha = 0.01$ par exemple) ne semble pas avoir de sens, encore moins être stable par rapport aux perturbations.

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3. Numériquement, une fois la décomposition algébrique effectuée, il faut connaître les fonctions spéciales de Mittag-Leffler dans tout le plan *complexe*, ce qui n'est pas une mince affaire, rien que dans le cas le plus simple où $\alpha = 0.5$, le comportement de la fonction d'erreur est rarement implémenté dans tout le plan complexe.

Questions ouvertes

1. Numériquement, si l'on veut introduire ces modèles fractionnaires dans des codes standard, il faudrait pouvoir d'une manière ou d'une autre les réécrire ou les approcher par des systèmes différentiels, ce qui est précisément l'objet des représentations diffusives.

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1. Numériquement, si l'on veut introduire ces modèles fractionnaires dans des codes standard, il faudrait pouvoir d'une manière ou d'une autre les réécrire ou les approcher par des systèmes différentiels, ce qui est précisément l'objet des représentations diffusives.
2. D'un point de vue modélisation également, les dérivateurs fractionnaires correspondent à des filtres passe-haut *idéaux* en $(i\omega)^\alpha$ sur toutes les fréquences, alors qu'en pratique, le comportement fractionnaire n'est souvent observé que sur une bande de fréquences utiles ; ainsi le transfert réel ou plus réaliste pourra être tronqué en fréquences, n'étant plus, à strictement parler, fractionnaire : dans ce cas, l'opérateur temporel correspondant sera *diffusif*.

Équations Différentielles Fractionnaires (II)

On considère la relation entrée u – sortie y :

$$\sum_{k=0}^{p} a_k \, D^{\alpha_k} y(t) = \sum_{l=0}^{q} b_l \, D^{\beta_l} u(t),$$

ou système fractionnaire linéaire à coefficients constants.
C'est un système pseudo-différentiel *causal*, dont le symbole est, par transformée de Laplace dans un demi-plan droit $\mathbb{C}_a^+$:

$$\mathcal{H}(s) = \frac{N(s)}{D(s)} \quad \text{avec} \quad \left| \begin{array}{lll} N(s) & \triangleq & \sum_{l=0}^{l=q} b_l \, s^{\beta_l}, \quad 0 < \beta_l < \beta_{l+1} \\ D(s) & \triangleq & \sum_{k=0}^{k=p} a_k \, s^{\alpha_k}, \quad 0 < \alpha_k < \alpha_{k+1} \end{array} \right.$$

Condition Nécessaire et Suffisante de Stabilité

Theorem

stabilité EBSB $\iff \exists M > 0, \quad |\mathcal{H}(s)| \leq M \quad \forall s, \Re e(s) \geq 0 .$

De plus, quand aucune simplification n'apparaît entre le numérateur et le dénominateur, (i.e. $\forall s, \Re e(s) \geq 0,$ $N(s) = 0 \Longrightarrow D(s) \neq 0$*), la condition de stabilité EBSB s'écrit :*

$$\textit{stabilité EBSB} \iff \begin{cases} \beta_q \leq \alpha_p \\ D(s) \neq 0 \quad \forall s, \Re e(s) \geq 0 \end{cases}$$

Ceci se fonde sur la finitude du nombre de pôles de $\mathcal{H}$ d'une part, et un nouveau résultat de structure d'autre part.

Résultat de Structure

Theorem

Lorsque $\beta_q < \alpha_p$*, la réponse impulsionnelle h du système est :*

$$h(t) = \sum_{k=1}^{K} \sum_{l=1}^{L_k} r_{k,l} \frac{1}{l!} t^{l-1} e^{s_k t} + \int_0^\infty e^{-\xi t} M(\mathrm{d}\xi) \qquad \text{pour } t > 0.$$

où les s_k *sont des pôles complexes dans* $\mathbb{C} \backslash \mathbb{R}^-$ *et M est en général une distribution tempérée causale.*
Dans le cas particulier où M est une mesure, absolument continue par rapport à la mesure de Lebesgue sur $\mathbb{R}^+$ *selon* $M(\mathrm{d}\xi) = \mu(\xi)\,\mathrm{d}\xi$*, la densité* μ *est donnée par :*

$$\mu(\xi) = \frac{1}{\pi} \frac{\sum_{k=0}^{p} \sum_{l=0}^{q} a_k b_l \sin((\alpha_k - \beta_l)\pi)\, \xi^{\alpha_k + \beta_l}}{\sum_{k=0}^{p} a_k^2\, \xi^{2\alpha_k} + \sum_{0 \le k < l \le p} 2 a_k a_l \cos((\alpha_k - \alpha_l)\pi)\, \xi^{\alpha_k + \alpha_l}};$$

Comportement Asymptotique

Lemma (Watson)

Si au voisinage de $\xi = 0^+$, $\exists -1 < \gamma_1 < \gamma_m < \gamma_{m+1}$, *tels que :*

$$\mu(\xi) = \sum_{m=1}^{M-1} \mu_m \frac{\xi^{\gamma_m}}{\Gamma(1+\gamma_m)} + O(\xi^{\gamma_M}),$$

alors, le développement asymptotique de h_μ *en* $t = +\infty$ *est :*

$$h_\mu(t) \triangleq \int_0^\infty \mu(\xi)\, e^{-\xi t}\, d\xi = \sum_{m=1}^{M-1} \mu_m \frac{1}{t^{1+\gamma_m}} + O(t^{-1-\gamma_M}).$$

Theorem

Lorsque $\beta_q \leq \alpha_p$,

$$\text{stabilité EBSB} \iff \begin{cases} \forall k \in [1, K], \quad \Re e(s_k) < 0, \\ \gamma_1 > 0 \end{cases}$$

Représentation Diffusive standard : définitions

Soit M une mesure positive sur $\mathbb{R}^+$ satisfaisant la condition dite de well-posedness (WP) suivante :

$$c_M \triangleq \int_0^\infty \frac{\mathrm{d}M}{1+\xi} < +\infty\,.$$

- On définit le système dynamique d'entrée $u \in L^2(0,T)$, de sortie $y \in L^2(0,T)$ et d'état $\phi \in H_M = L^2(\mathbb{R}^+, \mathrm{d}M)$:

$$\begin{aligned} \partial_t \phi(\xi,t) &= -\xi\,\phi(\xi,t) + u(t); \quad \phi(\xi,0) = 0, \quad \forall \xi \in \mathbb{R}^+, \\ y(t) &= \int_0^{+\infty} \phi(\xi,t)\,\mathrm{d}M(\xi)\,. \end{aligned}$$

- Alors, $y = h_M \star u$ où la réponse impulsionnelle s'écrit $h_M(t) = \int_0^\infty e^{-\xi\,t}\,\mathrm{d}M(\xi)$ pour $t > 0$.
- La fonction de transfert est $\mathcal{H}_M(s) = \int_0^\infty \frac{\mathrm{d}M(\xi)}{s+\xi}$, dans $\mathbb{C}_0^+$.

Représentation Diffusive standard : bilan d'énergie

Le bilan d' énergie suivant est vérifié, $\forall T > 0$:

$$\int_0^T u(t)\, y(t)\, dt = \frac{1}{2} \int_0^{+\infty} \phi(\xi, T)^2 \, \mathrm{d}M + \int_0^T \int_0^{+\infty} \xi \, \phi(\xi, t)^2 \, \mathrm{d}M \, \mathrm{d}t \ ,$$

où l'on peut décomposer le membre de droite en deux termes :

- une fonction de *storage*, au sens de Willems, évaluée à l'instant T seulement, $E_\phi(T) := \frac{1}{2} \, \|\phi(T)\|^2_{H_M}$,
- une énergie résiduelle dissipée sur tout l'intervalle $(0, T)$.

Exemple : $M_\beta(\mathrm{d}\xi) \triangleq \frac{\sin \beta\pi}{\pi} \, \xi^{-\beta} \, \mathrm{d}\xi$ pour $0 < \Re e(\beta) < 1$ vérifie la condition (WP), ce qui permet d'écrire une *réalisation diagonale* de l'*opérateur d'intégration fractionnaire* d'ordre β, dont la fonction de tranfert vaut $\mathcal{H}_\beta(s) = s^{-\beta}$.

Note : Les RD standard appartiennent à la classe des *well-posed systems*

Examples of diffusive representations

1. In the fields of viscoelasticity, the so-called memory variables are another name for the diffusive variables; thermodynamical constraints are strongly linked to the positivity of the operator.
2. In the fields of electromagnetism, many ad-hoc models have been developped, such as:
   - Cole–Cole: $(1+(\tau s)^{\alpha})^{-1}$,
   - Davidson–Cole: $(1+\tau s)^{-\gamma}$,
   - Havriliak–Negami: $(1+(\tau s)^{\alpha})^{-\gamma}$.

   All of them are of diffusive nature, though not strictly speaking fractional.

Acknowledgement

Thank you for your attention.