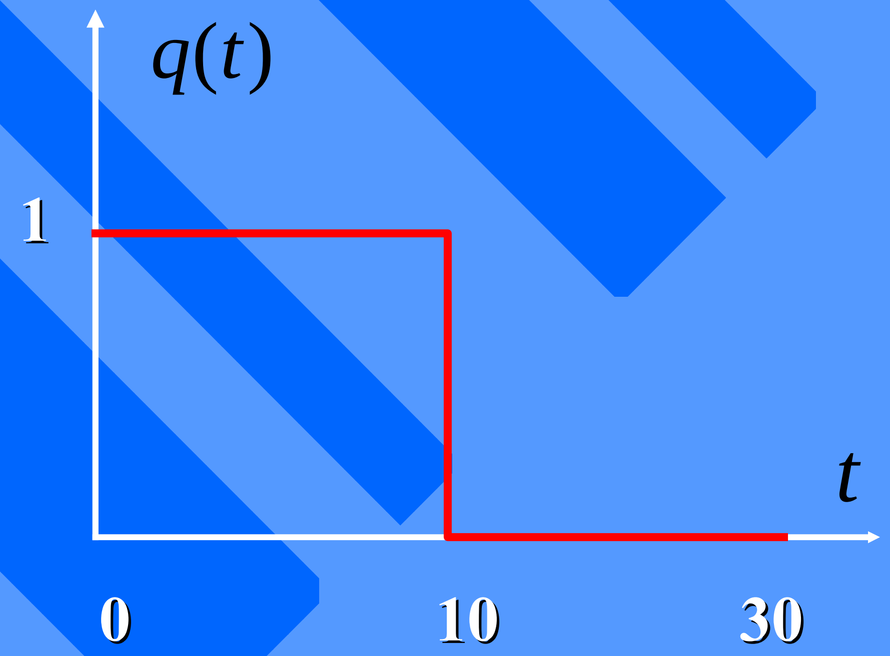
On the treatment of evolving interfaces:

Some numerical experiments on NEM and model reduction approaches

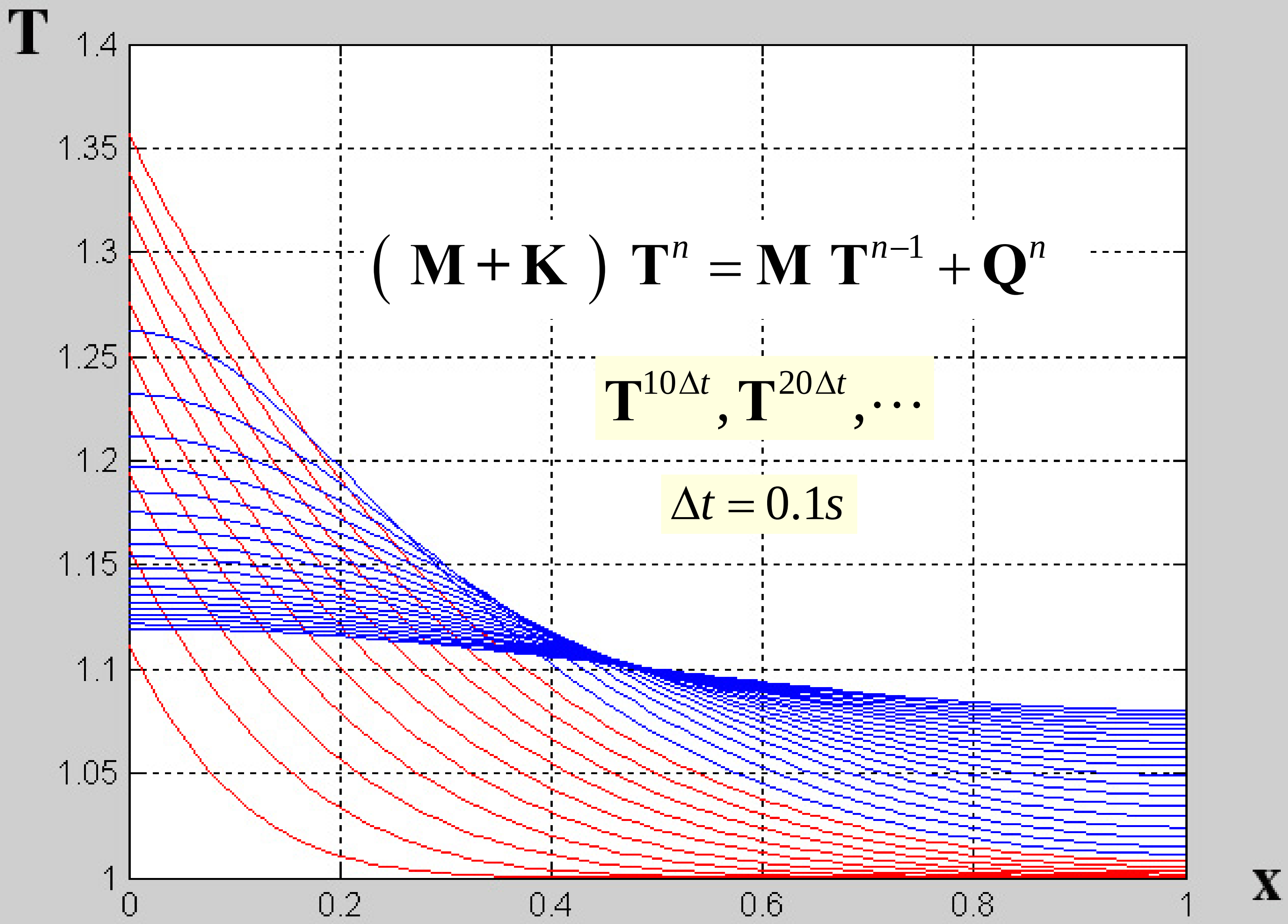
Introduction: Model reduction

A numerical example

$$\begin{cases} \dfrac{\partial T}{\partial t} = \alpha \dfrac{\partial^2 T}{\partial x^2} \qquad t \in \left]0,30\right], \; x \in \Omega = \left]0,1\right[\quad \alpha = 0.01 \\ T(x, t=0) = 1 \\ \left.\dfrac{\partial T}{\partial x}\right|_{x=0,t} = q(t) \\ \left.\dfrac{\partial T}{\partial x}\right|_{x=1,t} = 0 \end{cases}$$



$$\left(\mathbf{M} + \mathbf{K} \right) \mathbf{T}^{n} = \mathbf{M}\, \mathbf{T}^{n-1} + \mathbf{Q}^{n}$$


T
1.4
1.35
1.3
1.25
1.2
1.15
1.1
1.05
1
0
0.2
0.4
0.6
0.8
1
x
$(\mathbf{M}+\mathbf{K})\,\mathbf{T}^{n} = \mathbf{M}\,\mathbf{T}^{n-1} + \mathbf{Q}^{n}$
$\mathbf{T}^{10\Delta t}, \mathbf{T}^{20\Delta t}, \cdots$
$\Delta t = 0.1s$

Proper Orthogonal Decomposition

$$\mathbf{Q} = \begin{pmatrix} u_1^{10\Delta t} & u_1^{20\Delta t} & \cdots & u_1^{300\Delta t} \\ u_2^{10\Delta t} & u_2^{20\Delta t} & \cdots & u_2^{300\Delta t} \\ \vdots & \vdots & \ddots & \vdots \\ u_N^{10\Delta t} & u_N^{20\Delta t} & \cdots & u_N^{300\Delta t} \end{pmatrix}$$

⇒

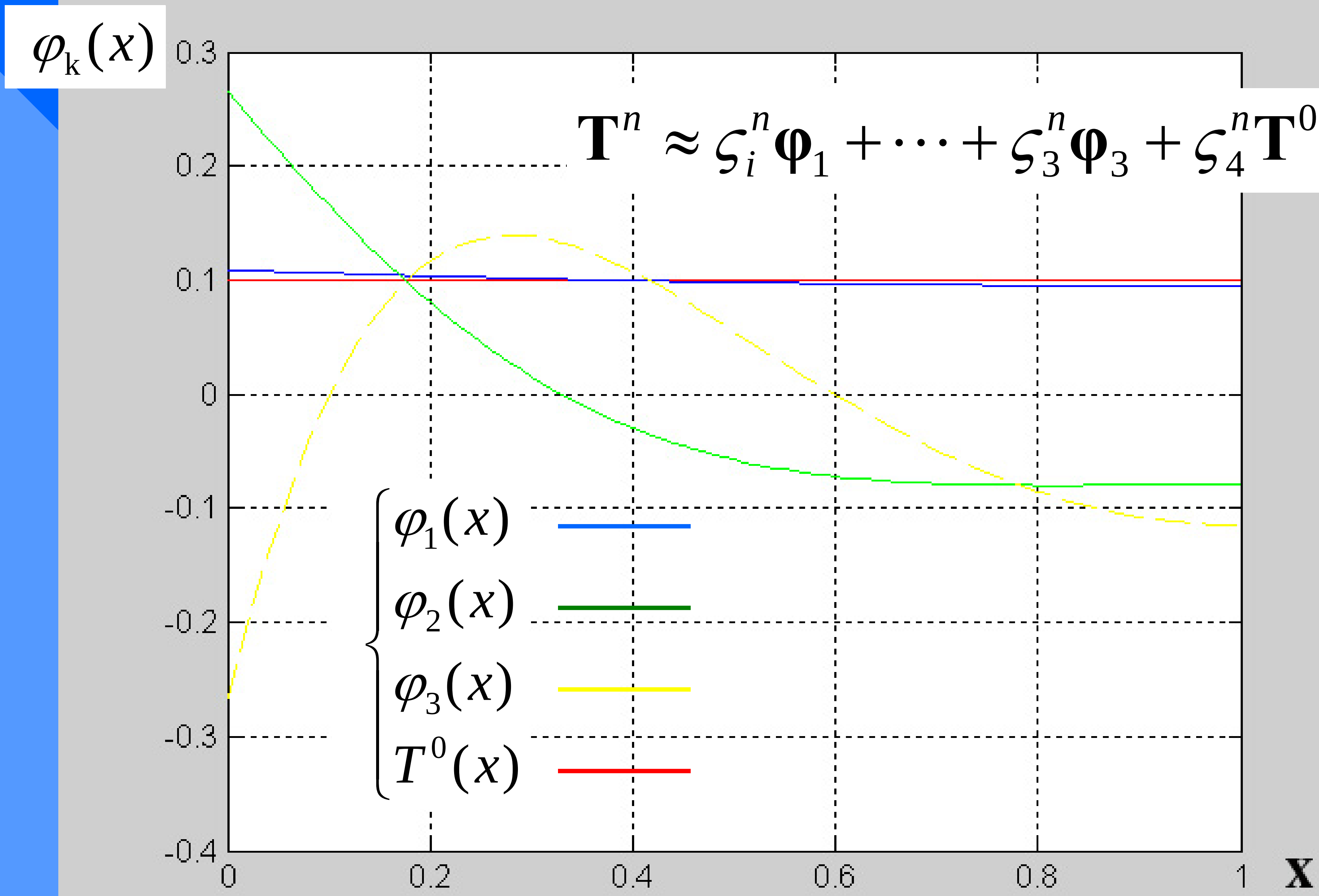
$$\left(\mathbf{Q}\mathbf{Q}^T\right)\boldsymbol{\varphi} = \alpha\boldsymbol{\varphi}$$

$$\left(\alpha_1, \boldsymbol{\varphi}_1\right) \ \ldots \ \left(\alpha_N, \boldsymbol{\varphi}_N\right)$$

$$\begin{cases} \alpha_1 = 1790 \\ \alpha_2 = 1.15 \\ \alpha_3 = 0.15 \\ \alpha_4 < 10^{-8}\alpha_1 \end{cases}$$

⇒

$$\begin{cases} \varphi_1(x) \\ \varphi_2(x) \\ \varphi_3(x) \\ T^0(x) \end{cases}$$



$\varphi_k(x)$
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0
0.2
0.4
0.6
0.8
1
X
$\mathbf{T}^n \approx \varsigma_i^n \boldsymbol{\varphi}_1 + \cdots + \varsigma_3^n \boldsymbol{\varphi}_3 + \varsigma_4^n \mathbf{T}^0$
$\varphi_1(x)$
$\varphi_2(x)$
$\varphi_3(x)$
$T^0(x)$

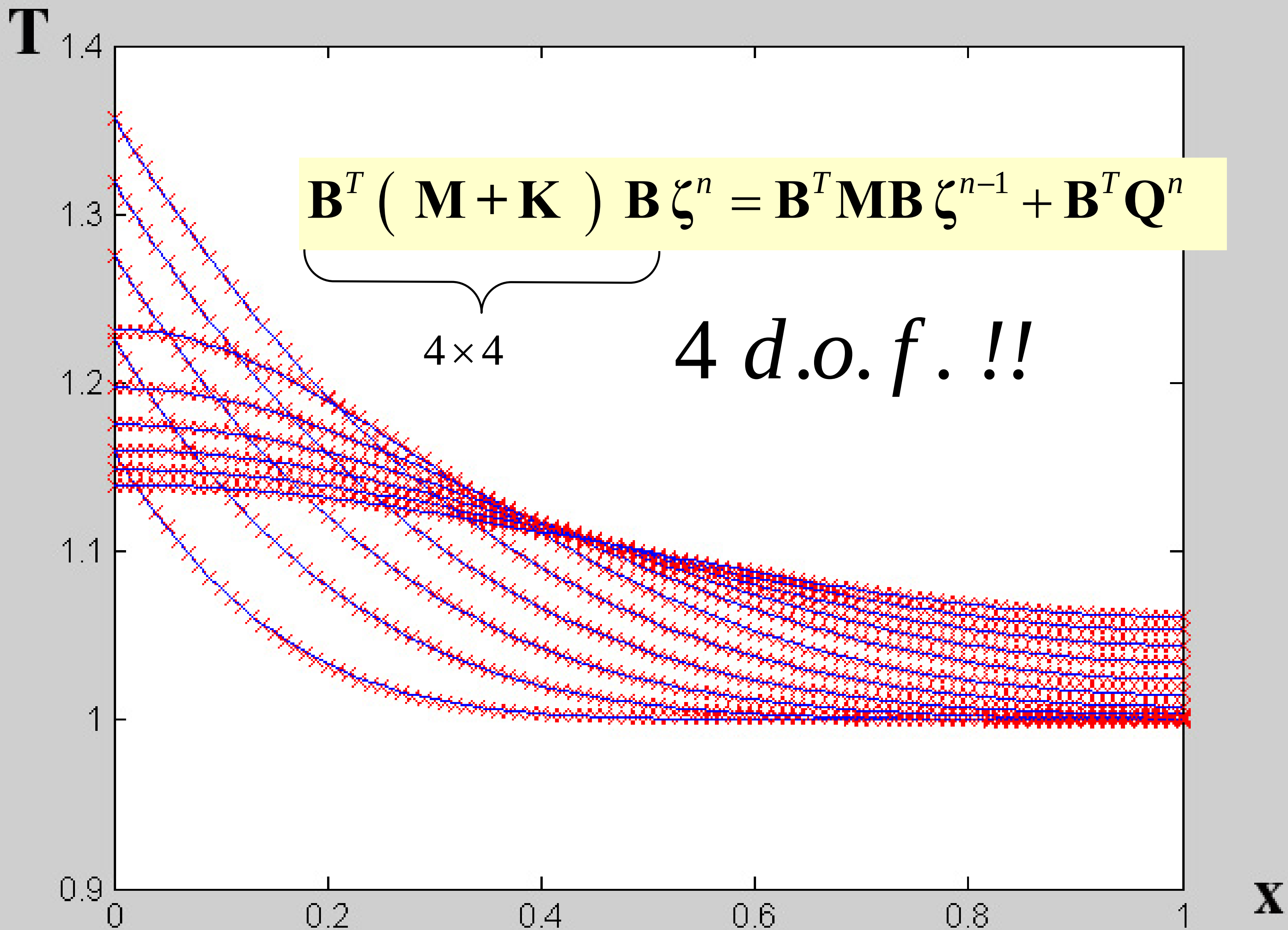
$$\mathbf{B} = \begin{pmatrix} T^0(x_1) & \varphi_1(x_1) & \varphi_2(x_1) & \varphi_3(x_1) \\ T^0(x_2) & \varphi_1(x_2) & \varphi_2(x_2) & \varphi_3(x_2) \\ \vdots & \vdots & \vdots & \vdots \\ T^0(x_N) & \varphi_1(x_N) & \varphi_2(x_N) & \varphi_3(x_N) \end{pmatrix}$$

$$\mathbf{T}^n = \mathbf{B}\,\zeta^n$$

Nx1

4x1

More than a significant reduction !!

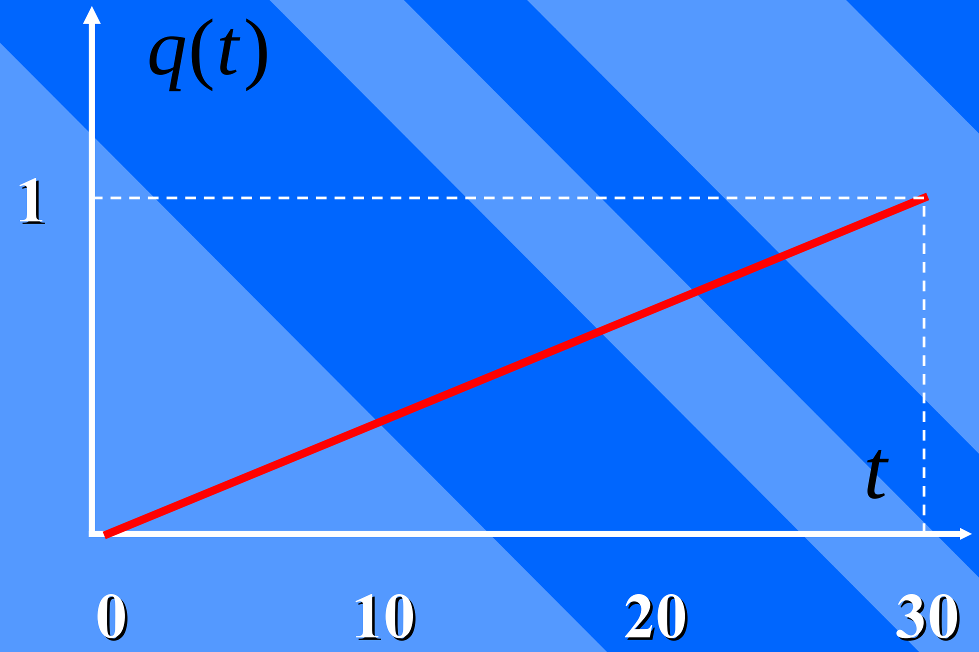


$$\mathbf{B}^T(\mathbf{M}+\mathbf{K})\mathbf{B}\,\zeta^n = \mathbf{B}^T\mathbf{M}\mathbf{B}\,\zeta^{n-1} + \mathbf{B}^T\mathbf{Q}^n$$

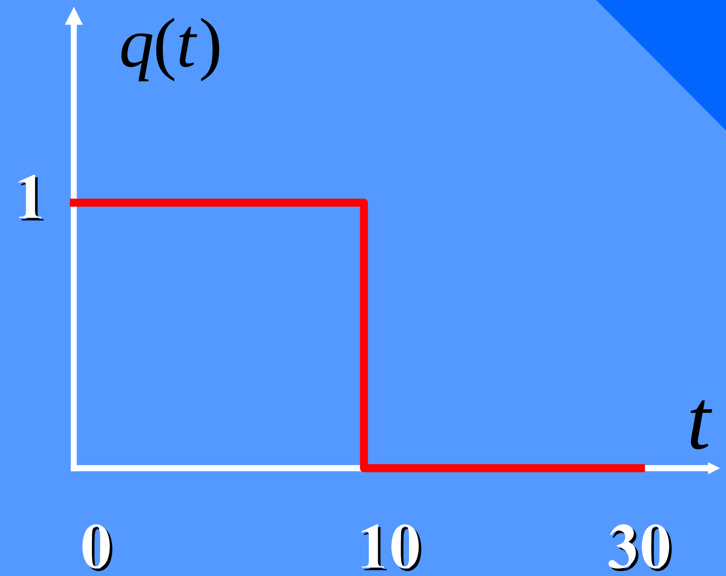
4×4

$4\ d.o.f.\ !!$

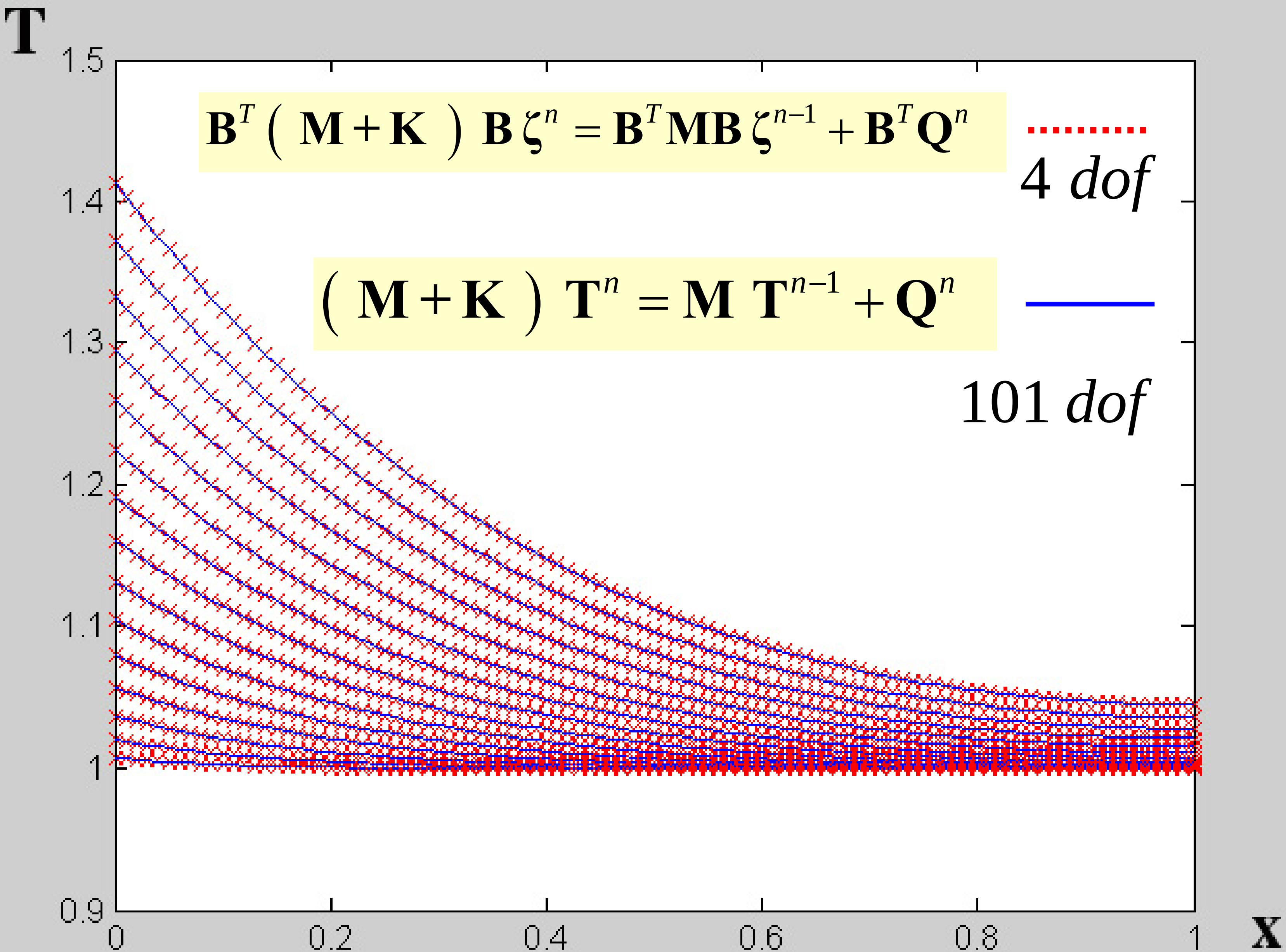
Solving « a similar » problem



with the reduced order approximation basis computed from the solution of the previous problem

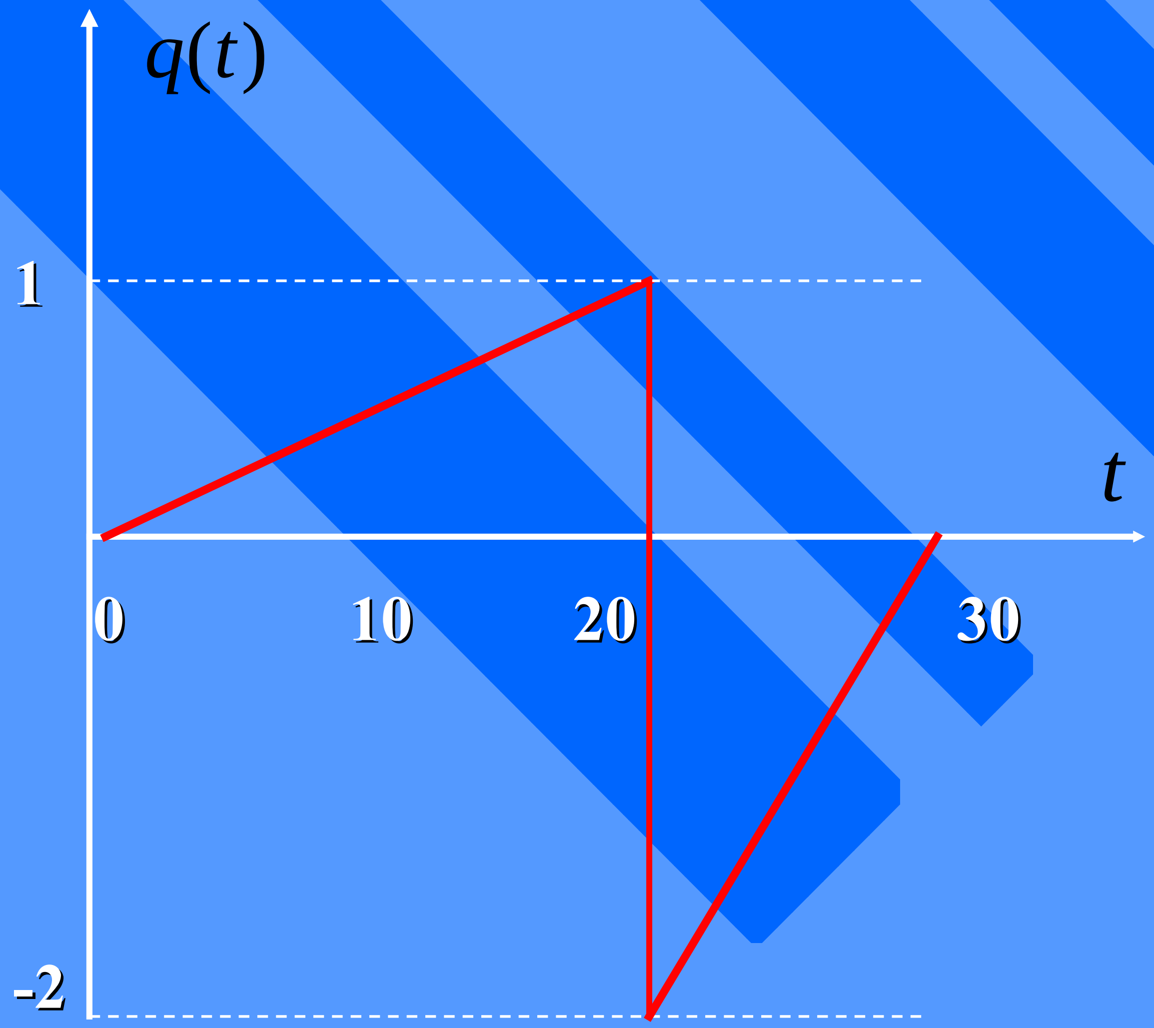


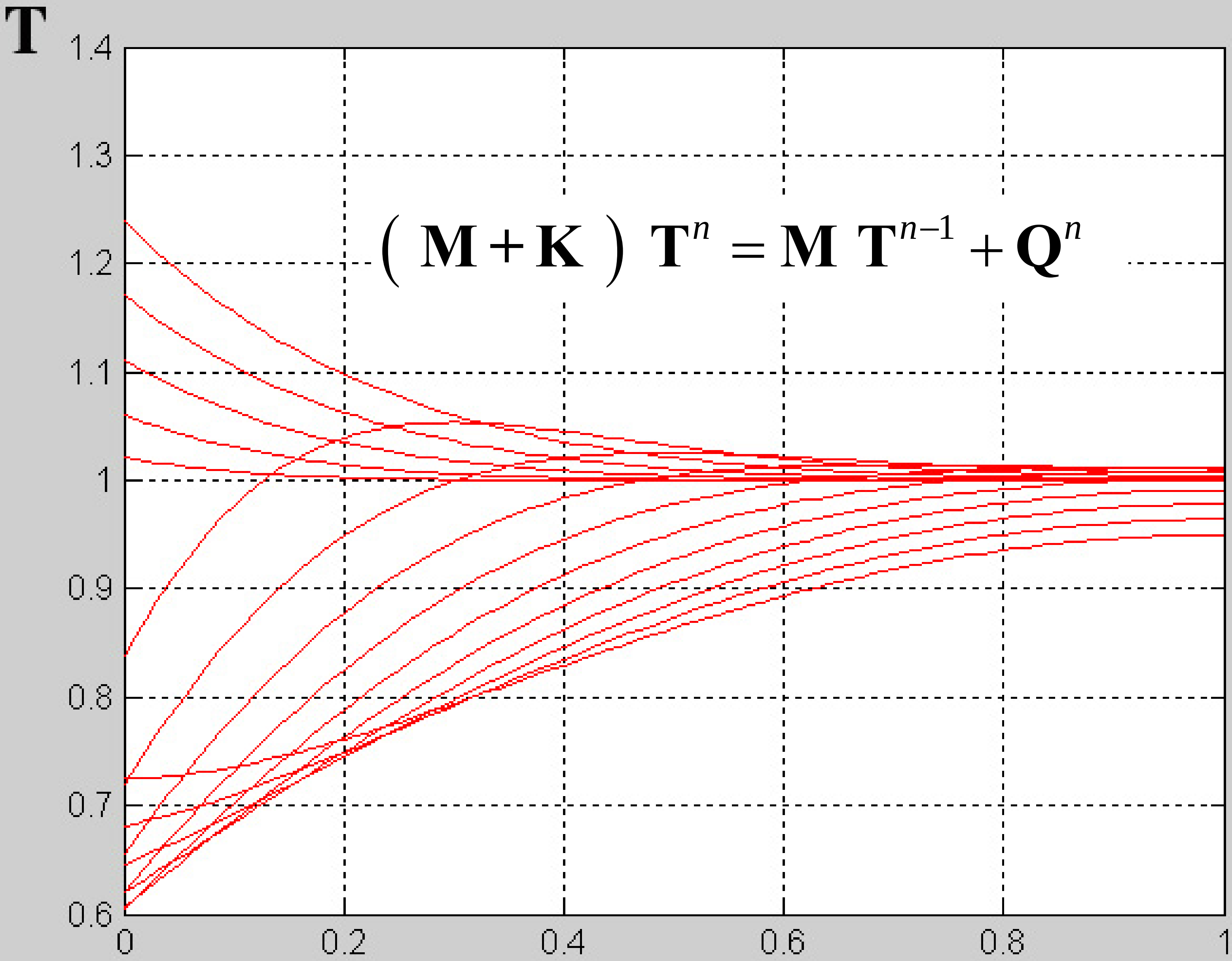
$$\mathbf{B} = \begin{pmatrix} T^0(x_1) & \varphi_1(x_1) & \varphi_2(x_1) & \varphi_3(x_1) \\ T^0(x_2) & \varphi_1(x_2) & \varphi_2(x_2) & \varphi_3(x_2) \\ \vdots & \vdots & \vdots & \vdots \\ T^0(x_N) & \varphi_1(x_N) & \varphi_2(x_N) & \varphi_3(x_N) \end{pmatrix}$$



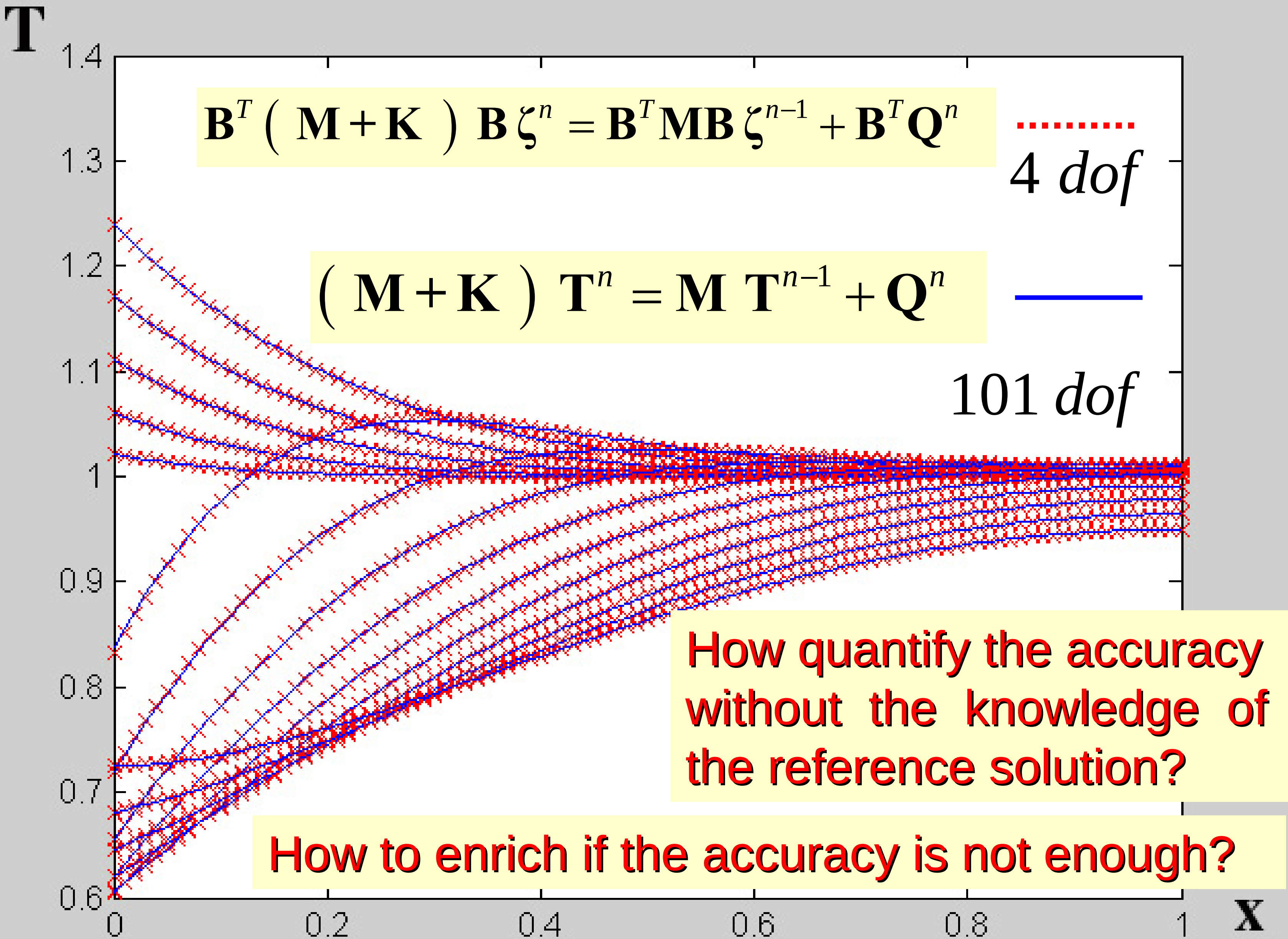
T
x
1.5
1.4
1.3
1.2
1.1
1
0.9
0
0.2
0.4
0.6
0.8
1
$\mathbf{B}^T(\mathbf{M}+\mathbf{K})\mathbf{B}\zeta^n = \mathbf{B}^T\mathbf{M}\mathbf{B}\zeta^{n-1} + \mathbf{B}^T\mathbf{Q}^n$
4 *dof*
$(\mathbf{M}+\mathbf{K})\mathbf{T}^n = \mathbf{M}\,\mathbf{T}^{n-1} + \mathbf{Q}^n$
101 *dof*

Evaluating accuracy and enriching the approximation basis





T
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0
0.2
0.4
0.6
0.8
1
$(\mathbf{M}+\mathbf{K})\,\mathbf{T}^{n} = \mathbf{M}\,\mathbf{T}^{n-1} + \mathbf{Q}^{n}$


T
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0
0.2
0.4
0.6
0.8
1
X
$\mathbf{B}^T(\mathbf{M}+\mathbf{K})\mathbf{B}\,\zeta^n = \mathbf{B}^T\mathbf{M}\mathbf{B}\,\zeta^{n-1} + \mathbf{B}^T\mathbf{Q}^n$
4 *dof*
$(\mathbf{M}+\mathbf{K})\,\mathbf{T}^n = \mathbf{M}\,\mathbf{T}^{n-1} + \mathbf{Q}^n$
101 *dof*
How quantify the accuracy without the knowledge of the reference solution?
How to enrich if the accuracy is not enough?

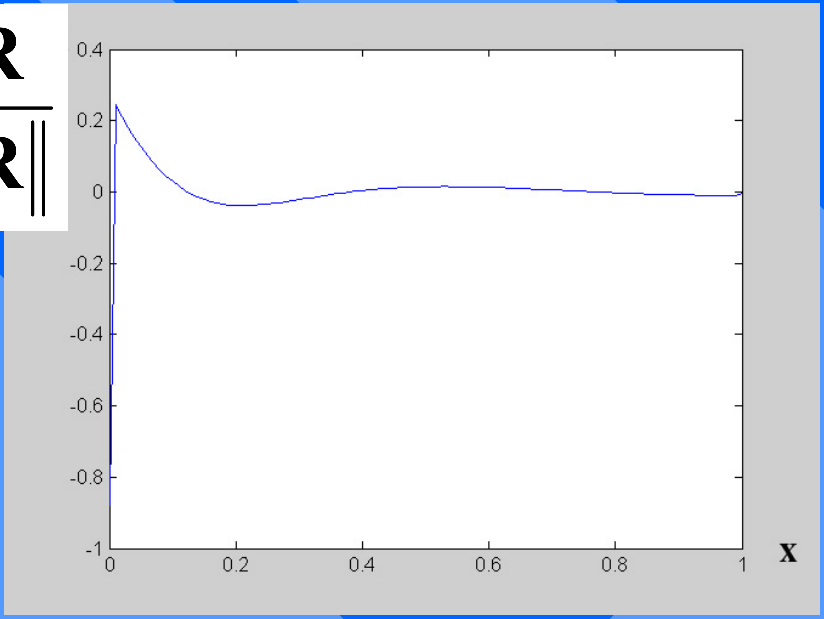
Control

$$\mathbf{B}^T (\mathbf{M}+\mathbf{K}) \mathbf{B} \zeta^n = \mathbf{B}^T\mathbf{M}\mathbf{B} \zeta^{n-1} + \mathbf{B}^T\mathbf{Q}^n$$

$\mathbf{T}^n$ $\mathbf{T}^{n-1}$

$$\mathbf{R} = (\mathbf{M}+\mathbf{K}) \mathbf{B} \zeta^n = \mathbf{M}\mathbf{B} \zeta^{n-1} + \mathbf{Q}^n \rightarrow \|\mathbf{R}\| = 2.6 \times 10^{-5}$$

$$\mathbf{R} = \frac{\mathbf{R}}{\|\mathbf{R}\|}$$



Enrichment

$$\mathbf{B} \leftarrow [\mathbf{B}, \mathbf{R}]$$

Time integration

$$\underline{\underline{A}}\,\underline{T}^{p} = \underline{F}^{p-1}$$

$$\underline{\underline{B}}$$

$$\underline{\underline{B}}^{T}\,\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{\zeta}^{p} = \underline{\underline{B}}^{T}\,\underline{F}^{p-1}$$

$$\underline{R} = \underline{\underline{A}}\,\underline{\underline{B}}\,\underline{\zeta}^{p} - \underline{F}^{p-1}$$

If $\left\|\underline{R}\right\| > \varepsilon$

$$\underline{\underline{B}} \leftarrow \left[\underline{\underline{B}}, \underline{R}\right]$$

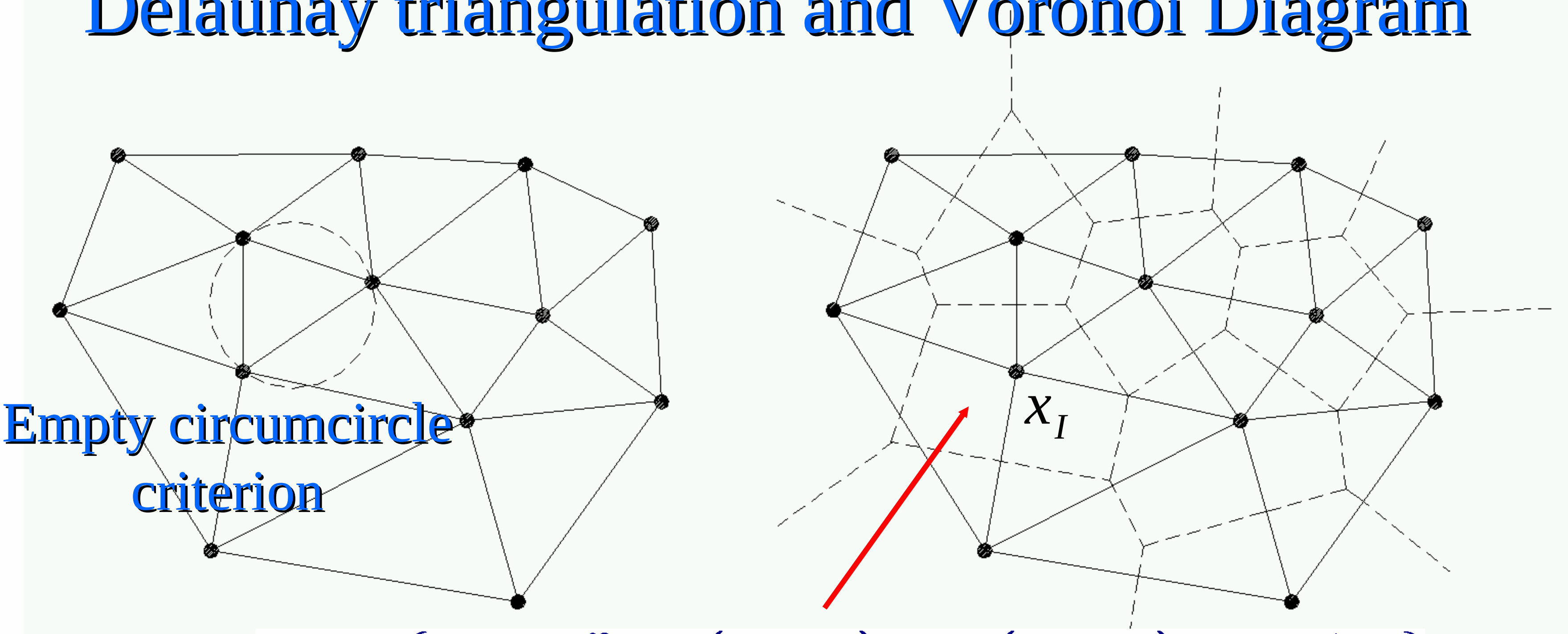
If $\left\|\underline{R}\right\| < \varepsilon$

$$\underline{\underline{B}}^{T}\,\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{\zeta}^{p} = \underline{\underline{B}}^{T}\,\underline{F}^{p-1}$$

Introduction: NEM approaches

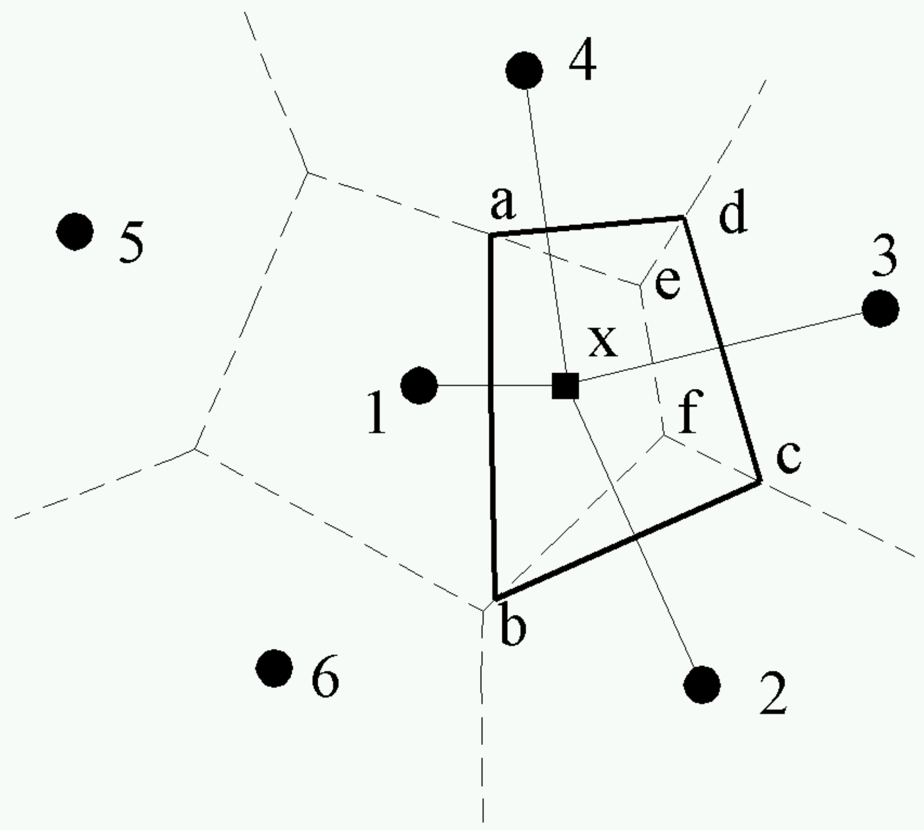
The Natural Element Method

Delaunay triangulation and Voronoï Diagram

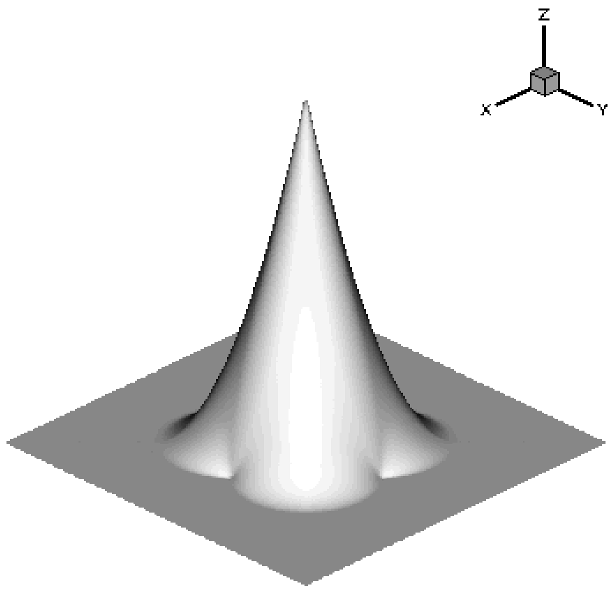


$$T_I = \{\boldsymbol{x} \in R^n : d(\boldsymbol{x}, \boldsymbol{x}_I) < d(\boldsymbol{x}, \boldsymbol{x}_J) \ \forall \ J \neq I\}$$

Sibson interpolation



$$N_1(x) = \frac{A_{abfe}}{A_{abcd}}$$



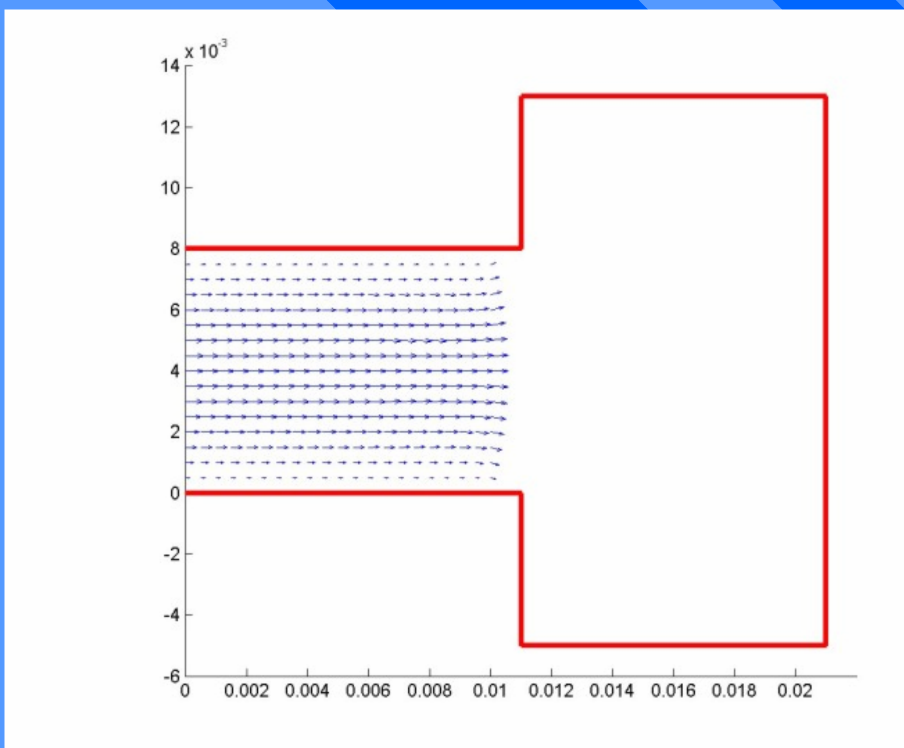
$$u(x) \approx u_1 N_1(x) + \cdots + u_N N_N(x)$$

Properties:

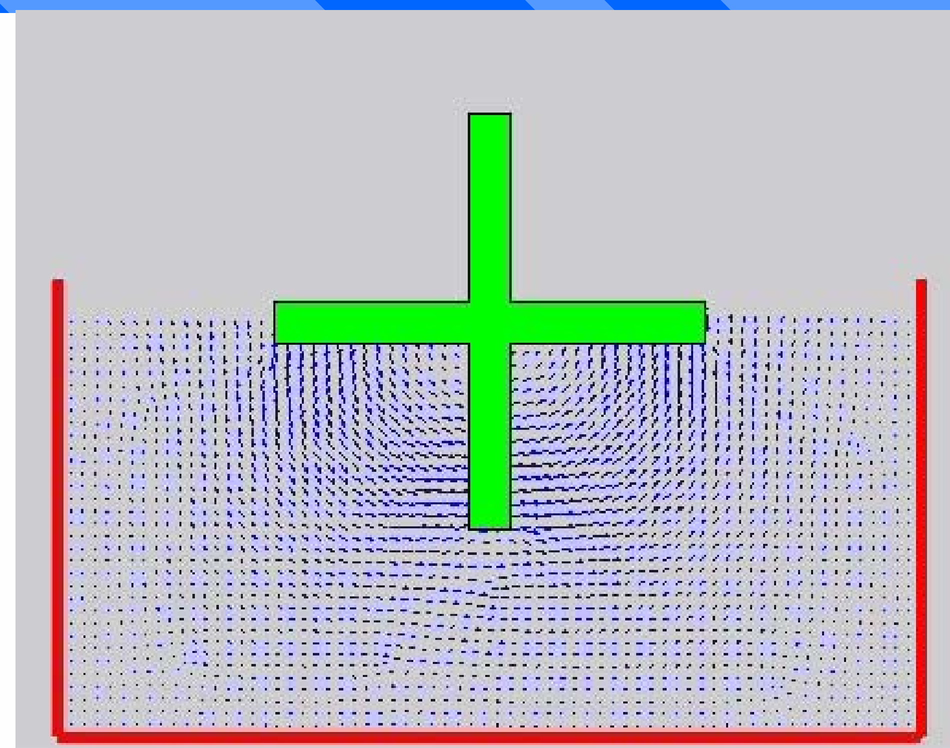
- ✓ Linear interpolation on the boundary (C or α-NEM)
- ✓ Linear consistency $\sum_i N_i(\underline{x})\underline{x}_i = \underline{x}$
- ✓ Partition of Unity $\sum_i N_i(\underline{x}) = 1 \Rightarrow$ PUM
- ✓ Better accuracy than linear FEM
- ✓ Meshless character: no sensibility to the distorsion of the *Delaunay* triangles
- ✓ The shape functions are C^1 everywhere except at the nodes positions.
- ✓ If one proceeds in an updated Lagrangian framework and the internal variables are defined at the nodes, remeshing, stabilization and field projections will be no more required.

Examples:

Viscoelastic flow

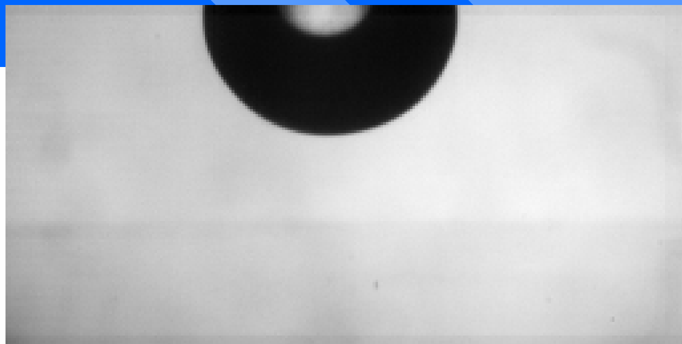


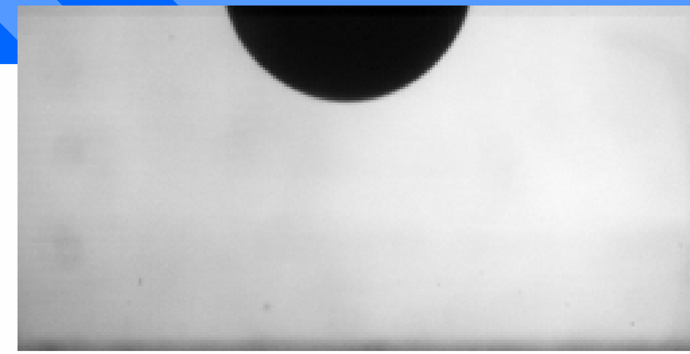
IFS

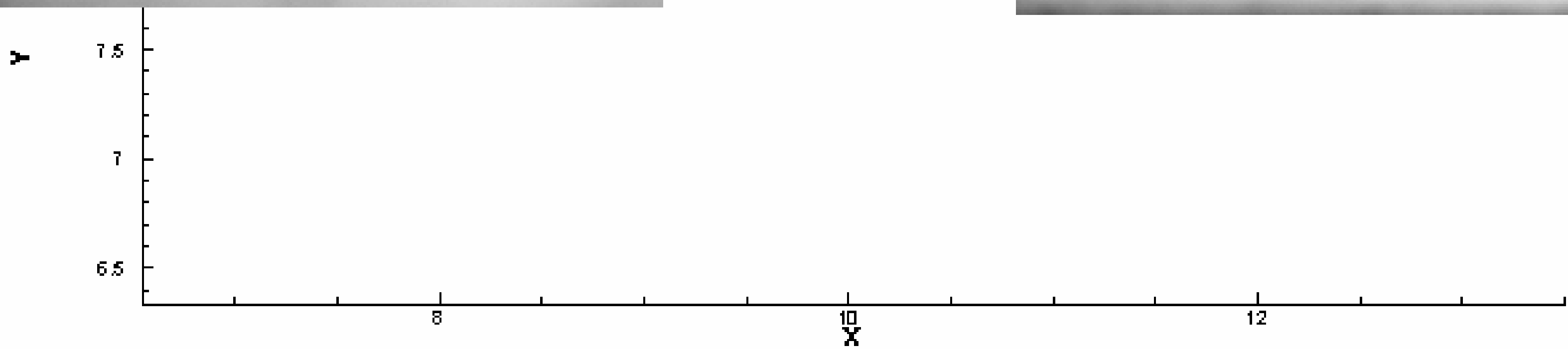


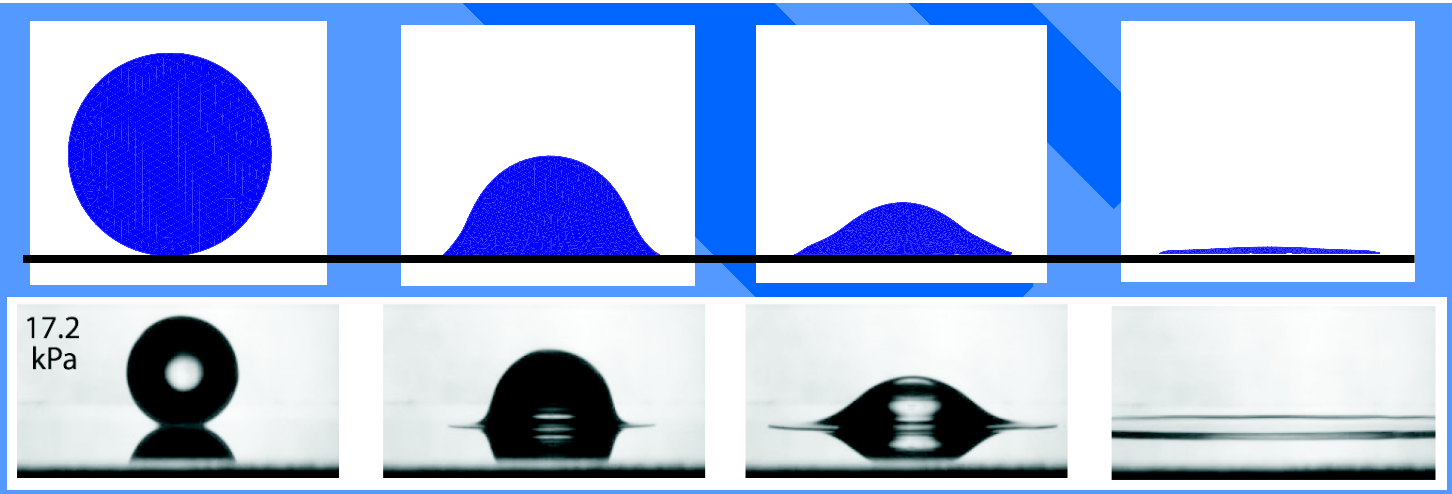
Atmospheric conditions

Vacuum conditions

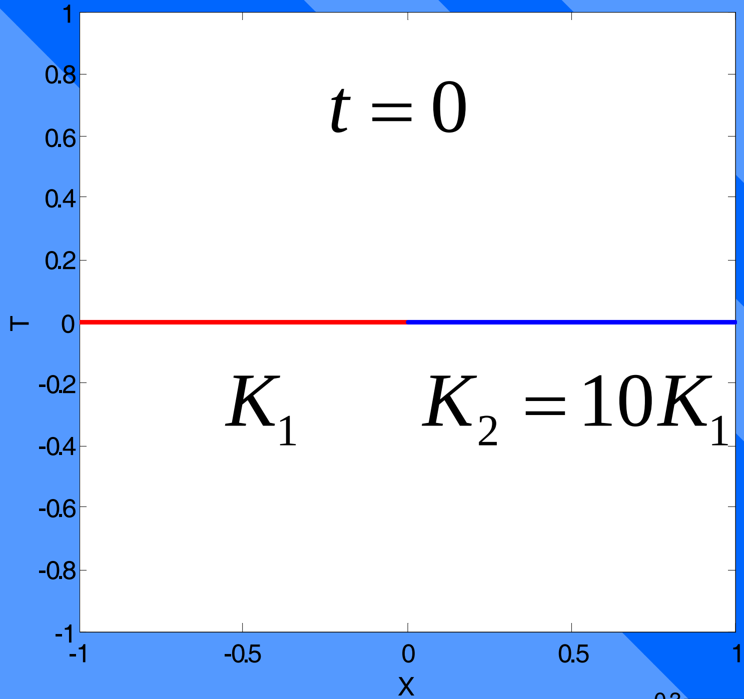






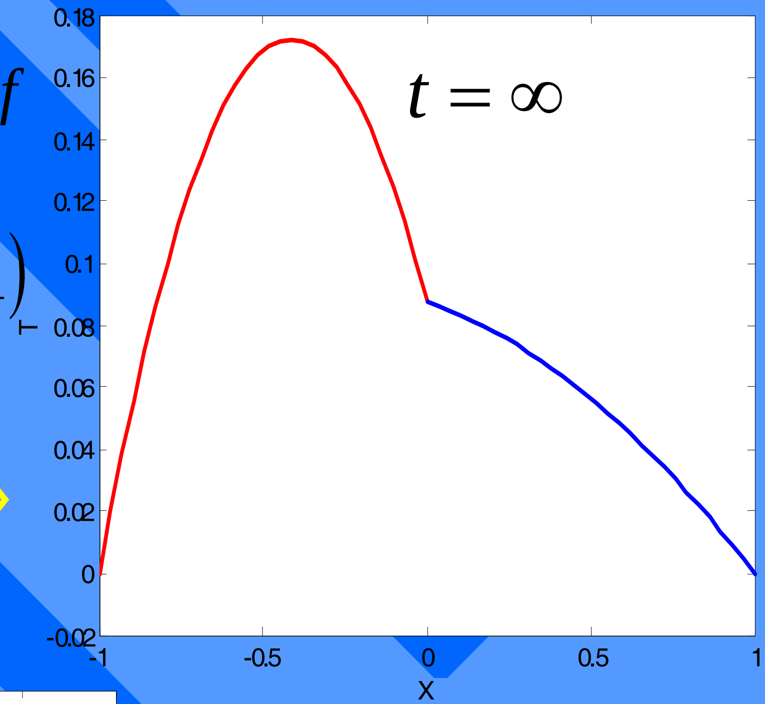


Treatment of Interfaces:
model reduction



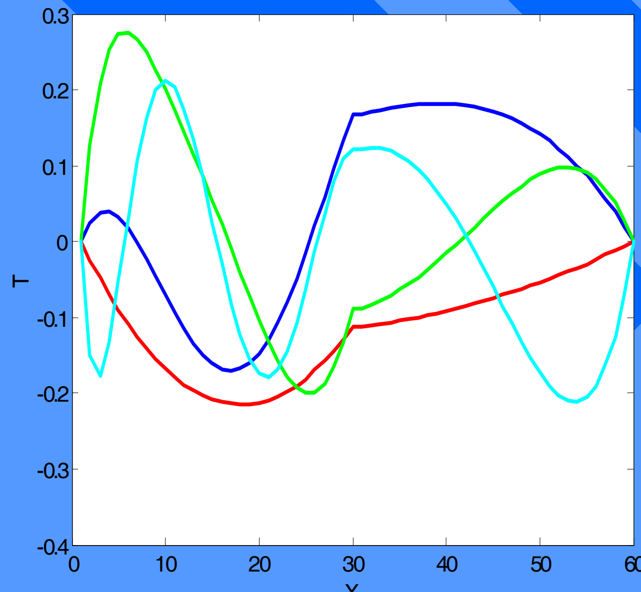
$$\frac{\partial T_i}{\partial t} = K_i \Delta T_i + f$$

$$\underline{T}^{p} = \underline{\underline{G}}\left(\underline{T}^{p-1} + \underline{F}\right)$$



4 dof

Eigenfunctions:

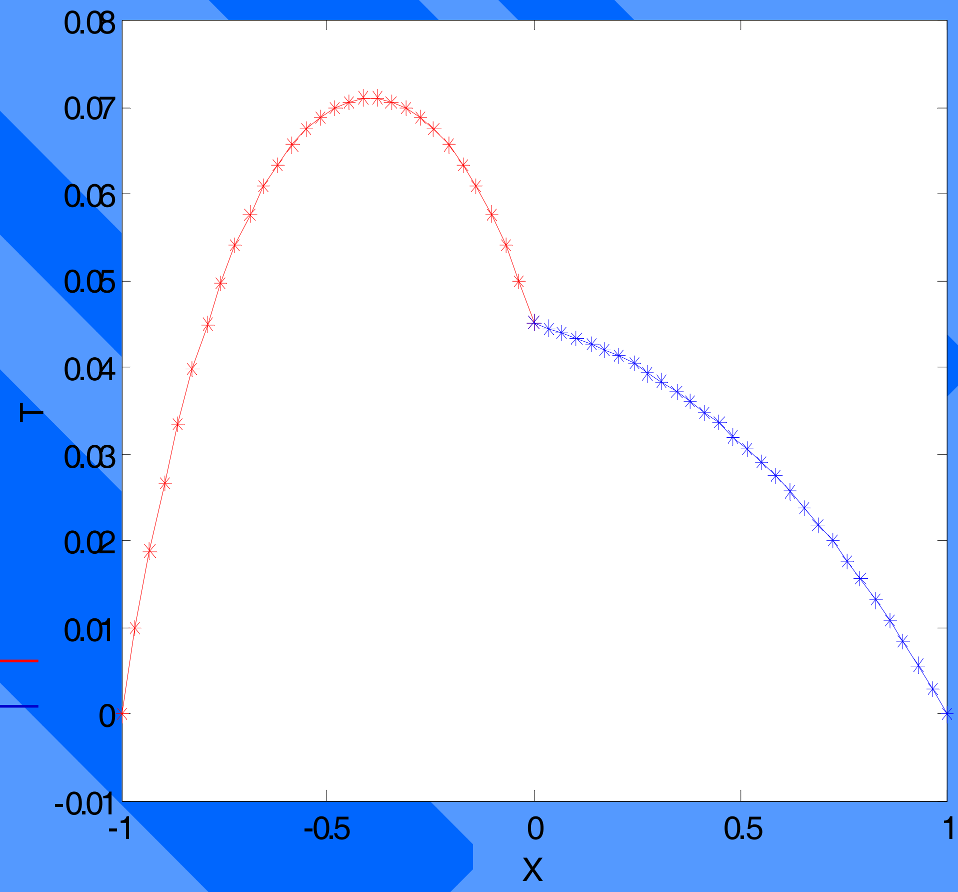


$$\underline{T}^{p} = \underline{\underline{B}}_{(N,4)}\,\underline{\xi}^{p}_{\ (4,1)}$$

$$\underline{T}^{p} = \underline{\underline{B}}_{(N,4)}\,\underline{\xi}^{p}_{(4,1)}$$

$$\underline{T}^{p} = \underline{\underline{G}}\left(\underline{T}^{p-1} + \underline{F}\right)$$

$$\underline{\xi}^{p} = \underline{\underline{g}}\left(\underline{\xi}^{p-1} + \underline{f}\right)$$



Treatment of Interfaces:
MEM approaches

Tracking or

Capturing

“Remeshing” or

Enforcing transmission conditions:

Reproduction conditions

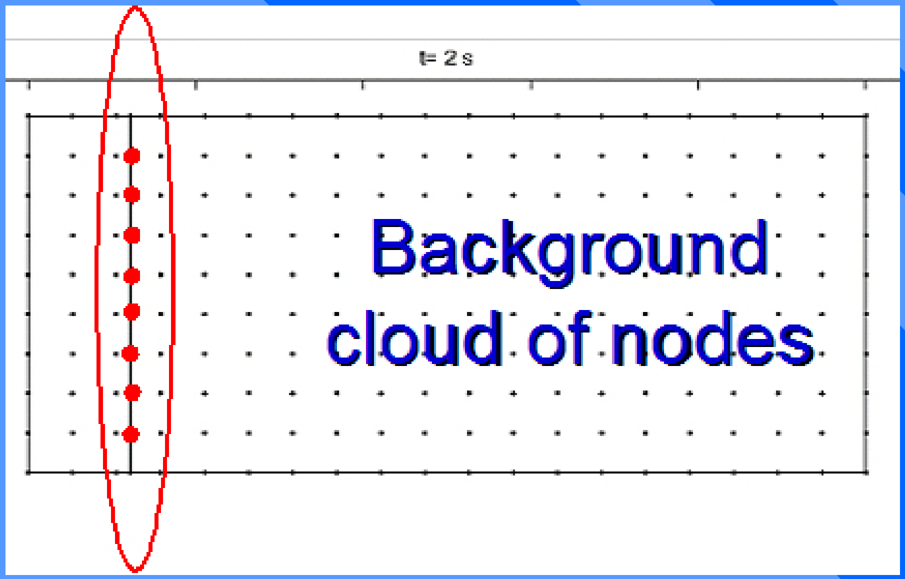
Enriching (X-…)

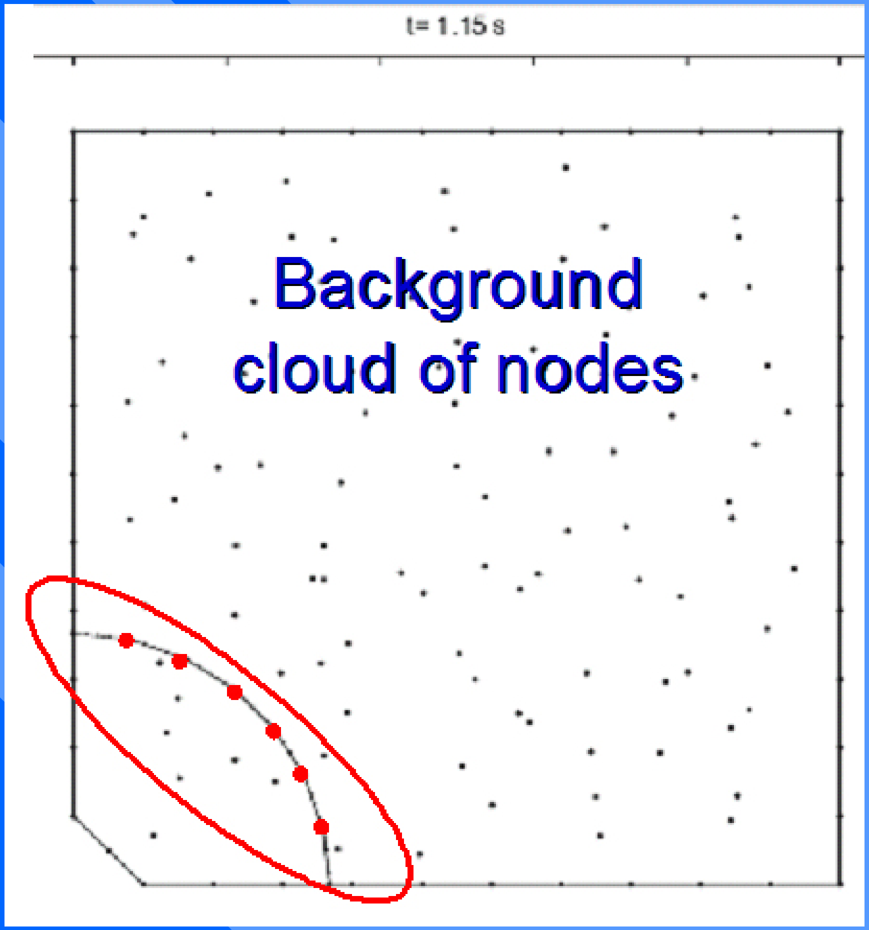
Why using NEM?

Markers nodes, OR

- level-set / background mesh

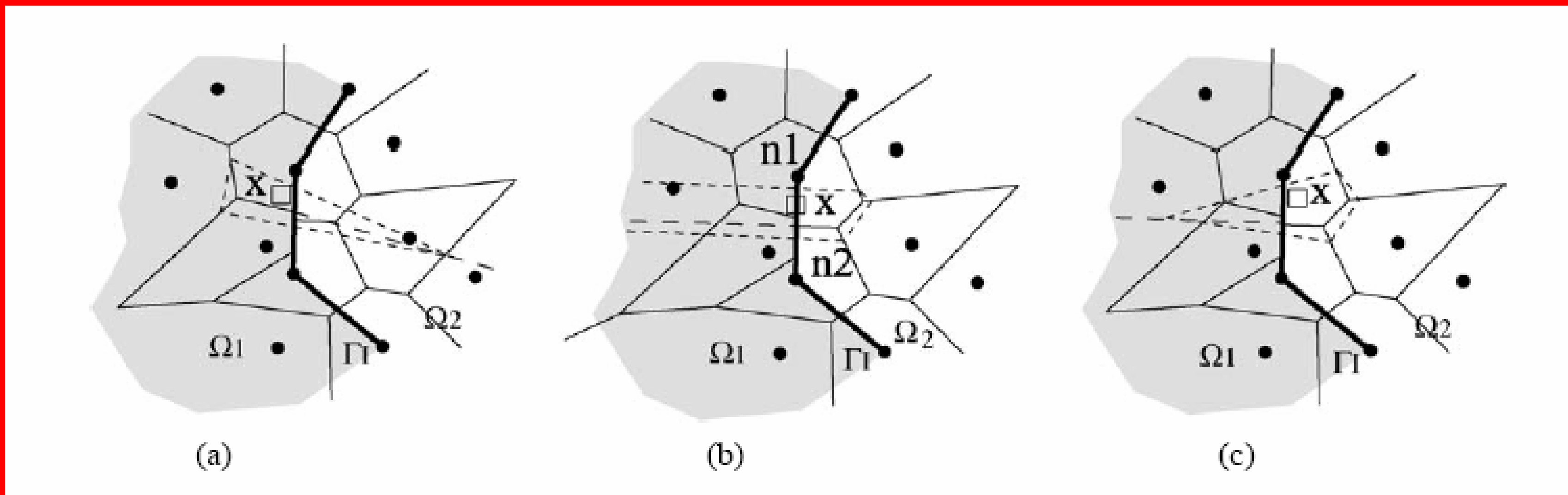
intersection points

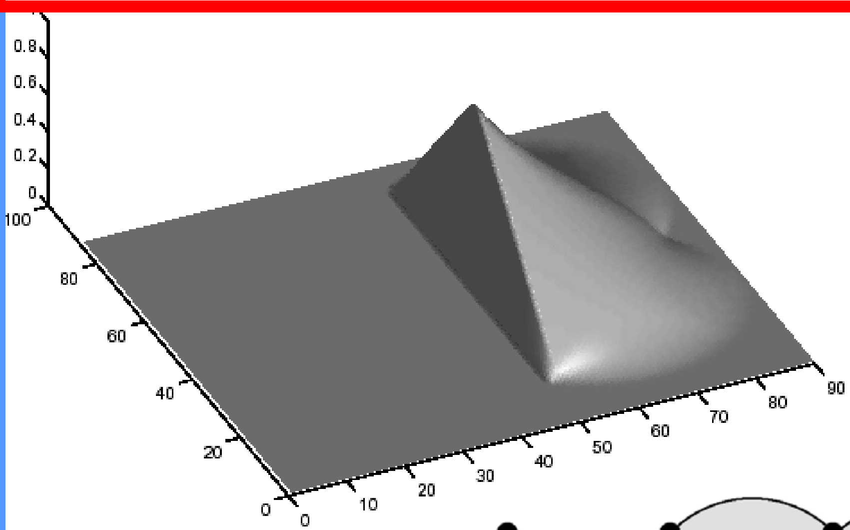


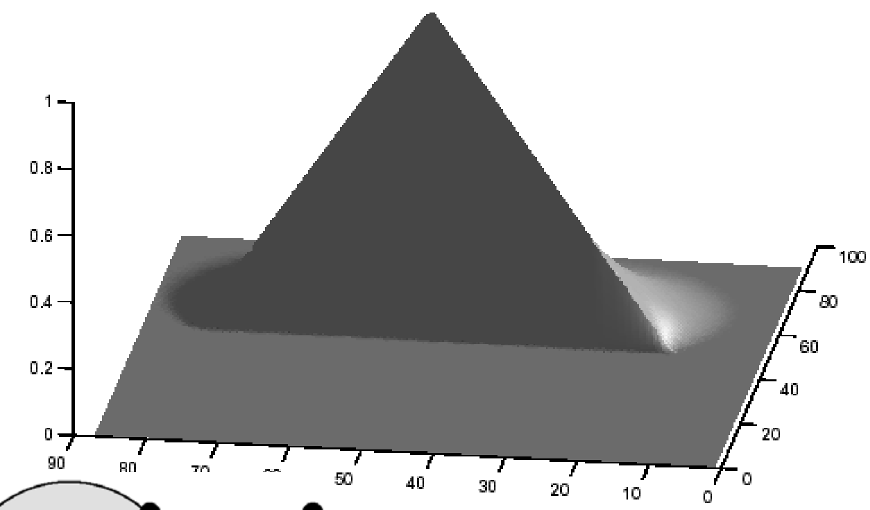


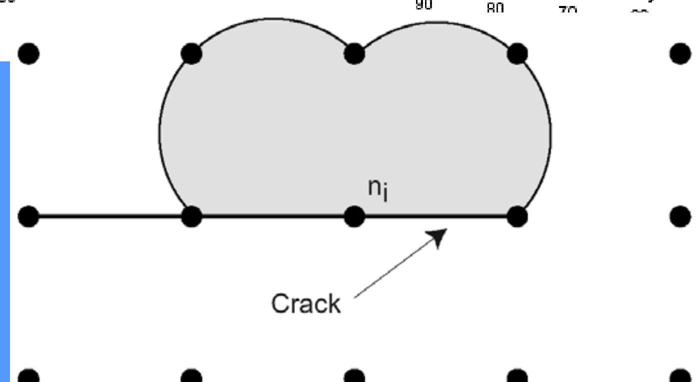
+ Update triangulation with respect to the updated interface

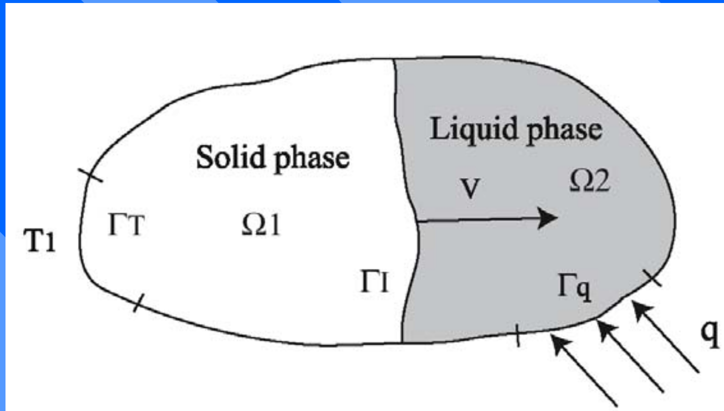
Delaunay triangles defined in the neighborhood of the interface are very distorted and we don't want to change the cloud of nodes


x
n1
n2
Ω1
Ω2
Γ1
(a)
(b)
(c)






n$_i$
Crack



$$\mathbf{V}(\mathbf{x} \in \Gamma_I(t)) = \frac{|[q]|}{L}\, \mathbf{n}_I(\mathbf{x})$$

$$|[q]| = \left(k_1 \nabla T|_{\Gamma_I^-(t)} - k_2 \nabla T|_{\Gamma_I^+(t)} \right)\, \mathbf{n}_I$$

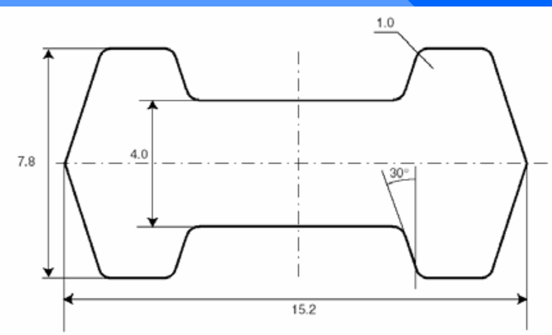
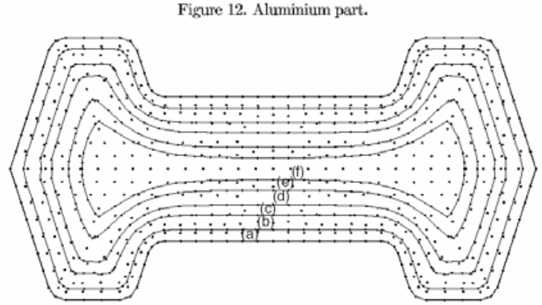
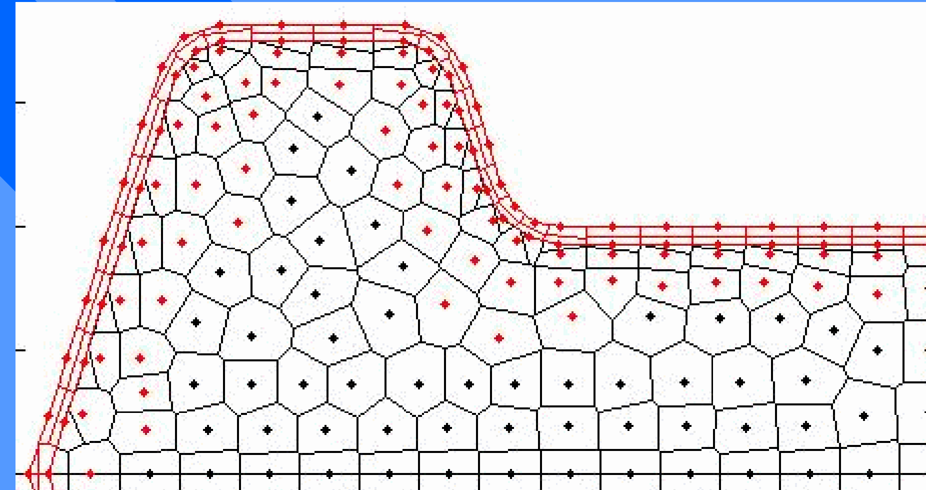
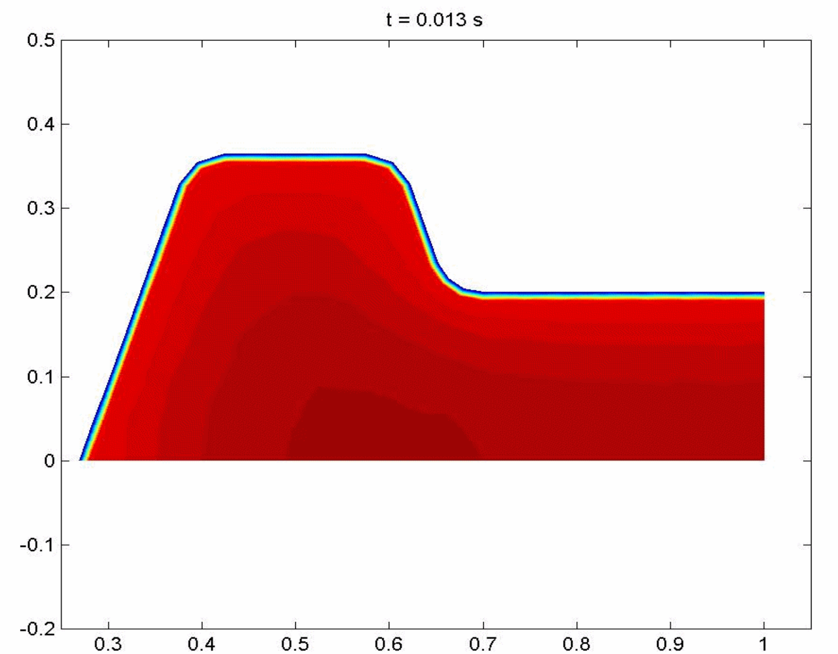
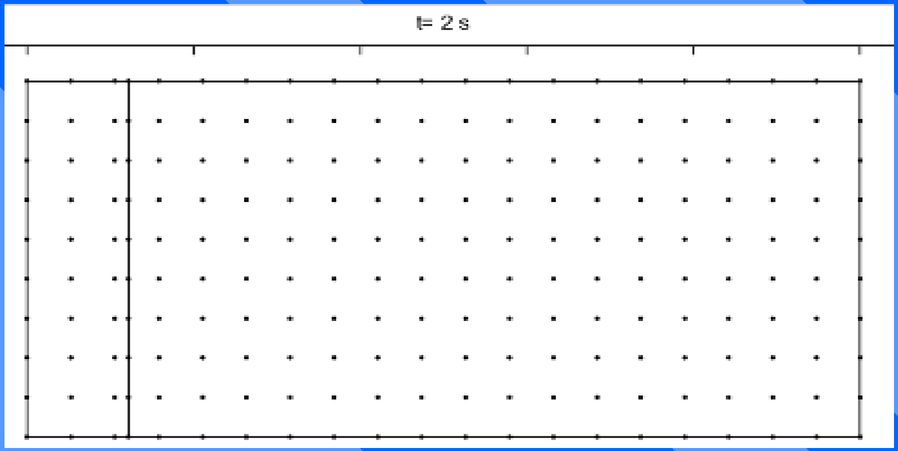


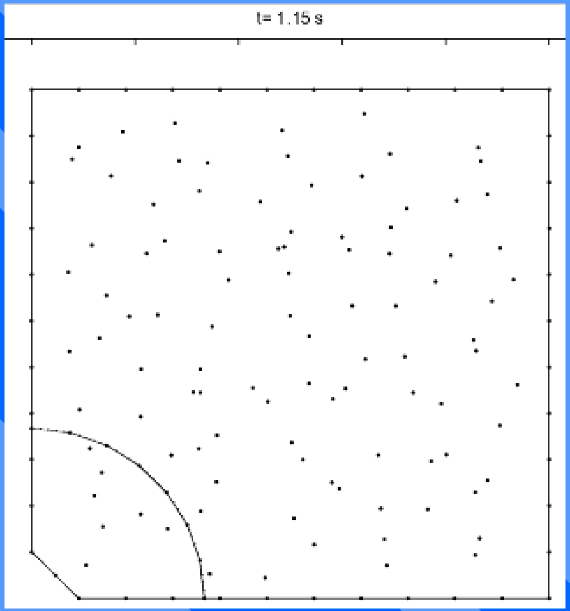
Figure 12. Aluminium part.

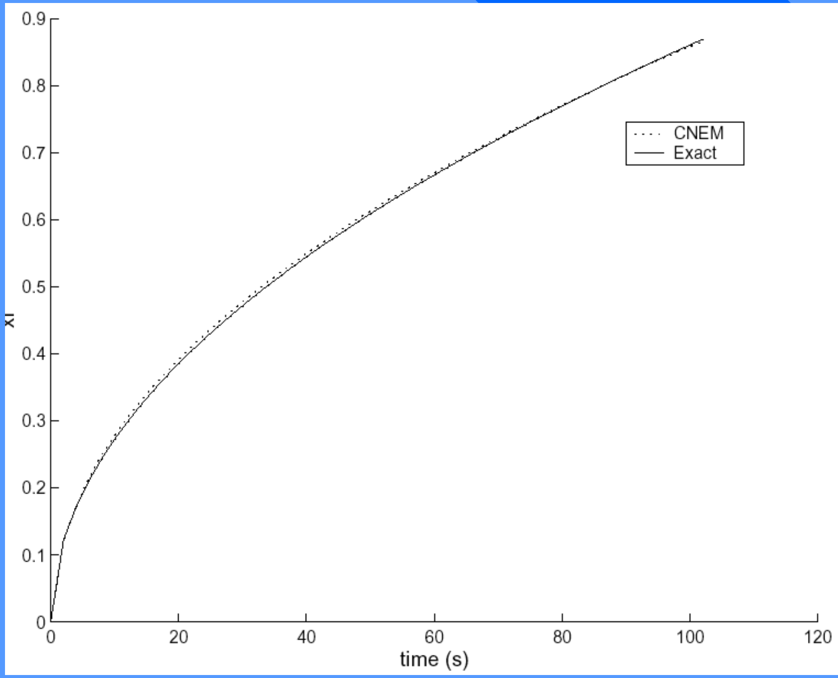


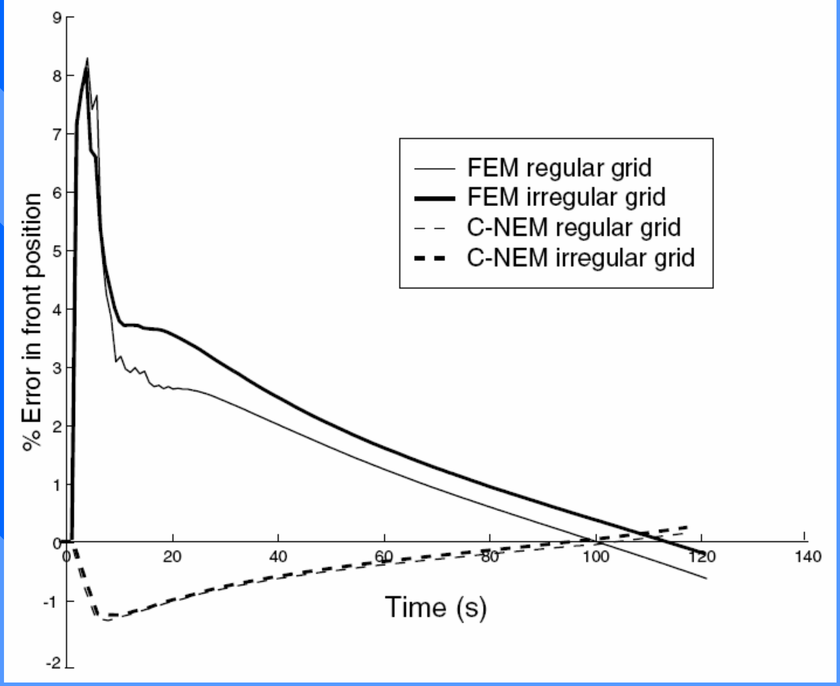





t= 2 s


t= 1.15 s


CNEM
Exact
time (s)


FEM regular grid
FEM irregular grid
C-NEM regular grid
C-NEM irregular grid
% Error in front position
Time (s)

Nice, BUT when nodes move, model reduction based on POD becomes a delicate procedure !

Enforcing transmission conditions

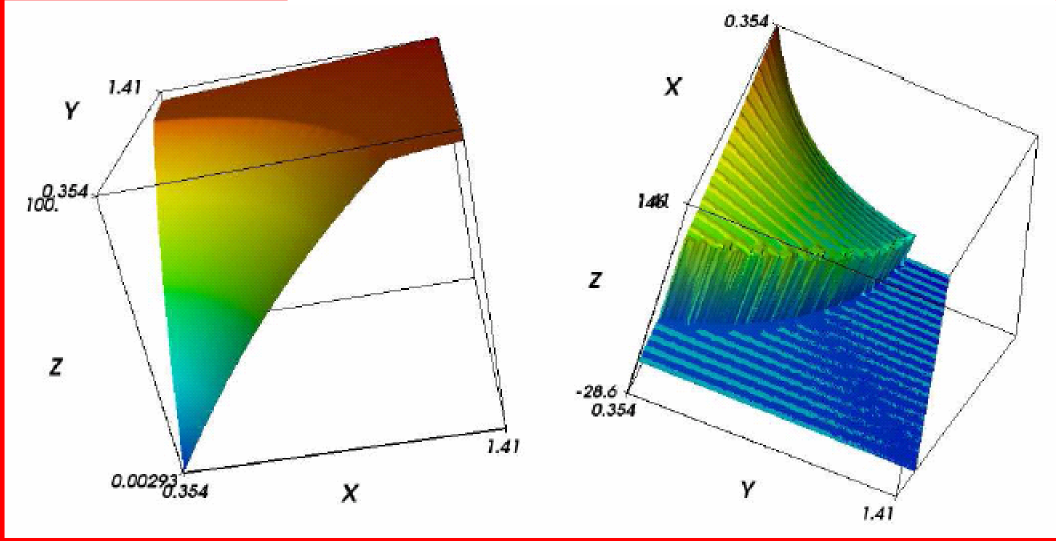
$$u^h(x) = \int_\Omega C(x, x-s)\, \phi_a(x-s, h)\, u(s)\, ds$$

C-NEM shape function

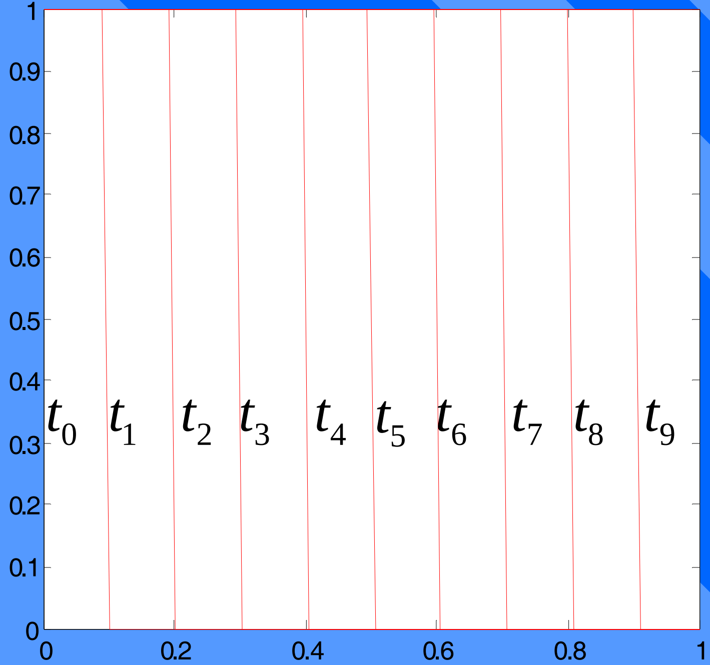
$$\begin{cases} \int_\Omega C(x,x-s)\ \phi_a(x-s,h)\ 1\ ds = 1 \\ \int_\Omega C(x,x-s)\ \phi_a(x-s,h)\ s\ ds = x \\ \vdots \\ \int_\Omega C(x,x-s)\ \phi_a(x-s,h)\ f^e(s)\ ds = f^e(x) \end{cases}$$

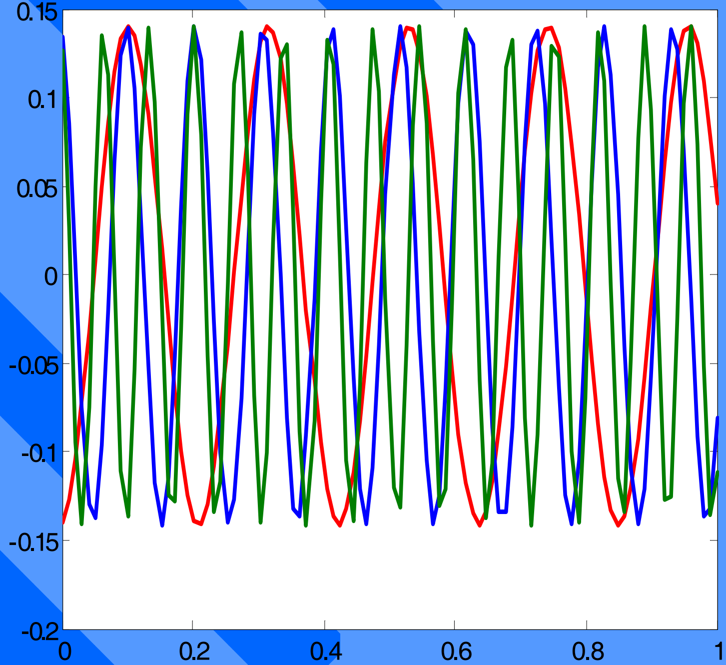
$$f^e(x) = \Theta(x) H_0(\Theta(x))$$

C-NEM shape functions verifying the transmission conditions.



The evolution of a characteristic function cannot be reduced in a POD sense !

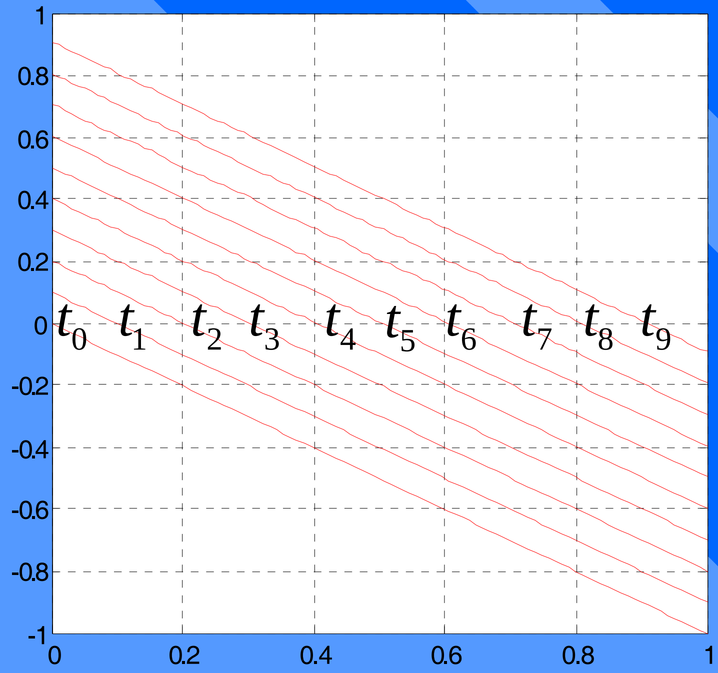


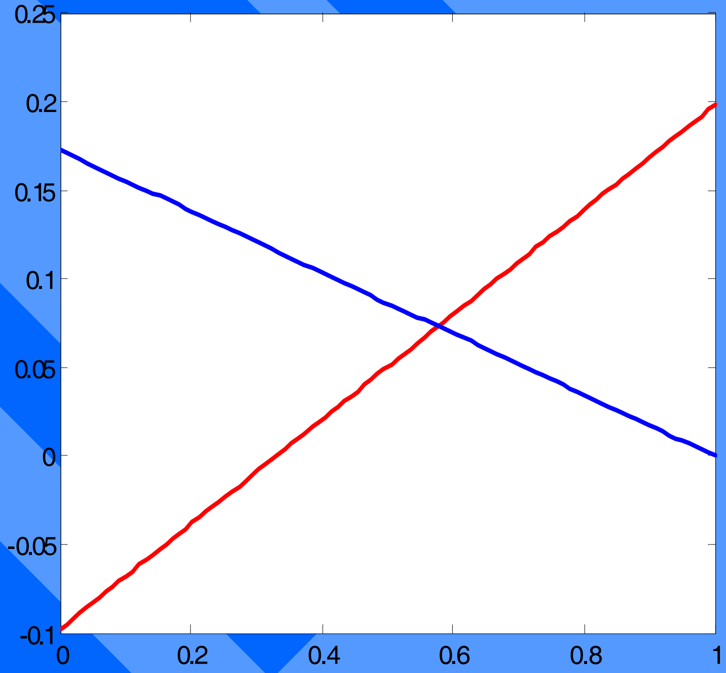


$$\frac{\lambda_1}{\lambda_N} \approx 10^4$$

⇒ Number of modes = Number of nodes !!!

BUT the evolution of the level set function can be also represented in a reduced approximation basis





$$\lambda_i = 0, \forall i > 2$$

Number of modes = 2

Now, the coupling seems possible!