Modélisation d'interfaces linéaires et non linéaires dans le cadre X-FEM

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OUTLINE

- PART 1 : X-FEM basics and continuous modeling of strain jumps
- PART 2 : Stability issues with stiff interfaces
- PART 3 : Applications

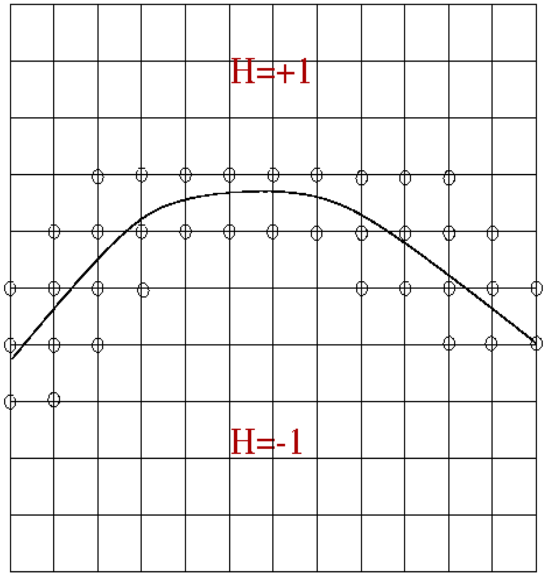
PART 1

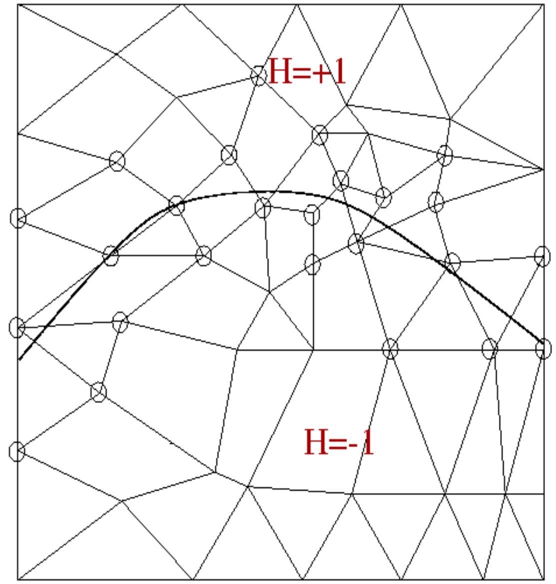
X-FEM BASICS

X-FEM Philosophy

- Use of the partition of unity to insert surfaces of discontinuity inside finite elements (disc. in a field, it's derivative or matter)
- Locate (and evolve) the surfaces on the mesh with level sets and use this function for to build the enrichment
- Manage the discontinuities with the same generaliy as FEM (dynamics, large deformations, mixed formulations, contact, ...) and all types of behaviour.
- Extend all the good convergence and stability properties of FEM

Enrichment to Model a Discontinuity in a Field





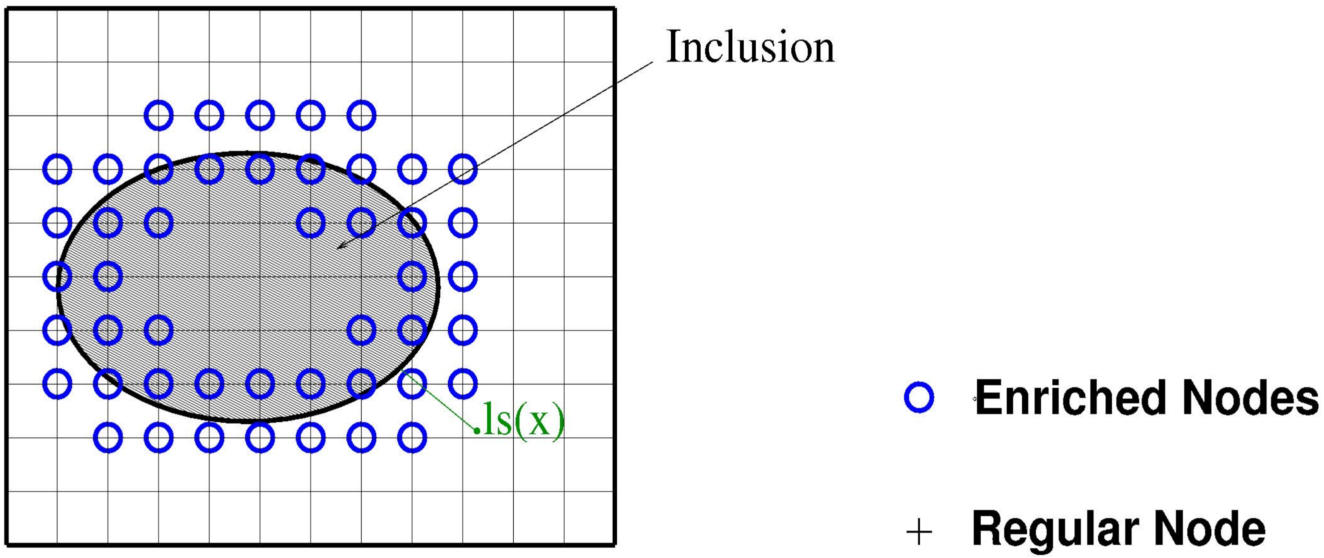
$$\mathbf{u}^h = \sum_{i \in I} \mathbf{u}_i \phi_i + \sum_{j \in J} \mathbf{b}_j \phi_j H(\mathbf{x})$$

Support Rule : A node is enriched if its support is split by the discontinuity.

Important : If the discontinuity is aligned with the mesh , we recover double-nodes : **X-FEM -> FEM**

(Moës, Dolbow,Belytschko 1999)

Enrichment For a Discontinuity in a Field Derivative



An node is enriched if its support is split by the material interface AND if at least one element of the support is split.

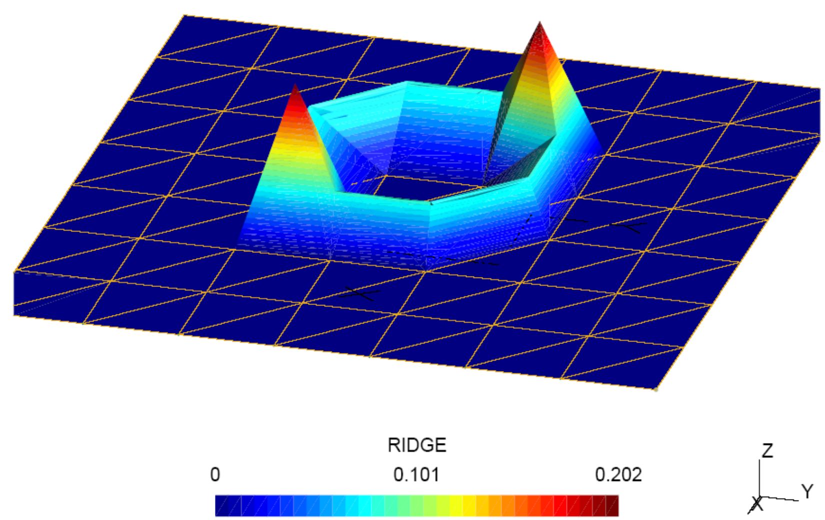
$$\mathbf{u}(\mathbf{x})|_{\Omega_e} = \sum_{\alpha=1}^{n_u} \mathbf{N}_{\mathbf{u}}^{\alpha} \left(u^{\alpha} + \sum_{\beta=1}^{n_e} a_{\beta}^{\alpha} \, \phi_{\beta}^{u}(\mathbf{x}) \right)$$

Ridge enrichment function (interpolant of the absolute level set nodal values minus absolute value of the interpolant)

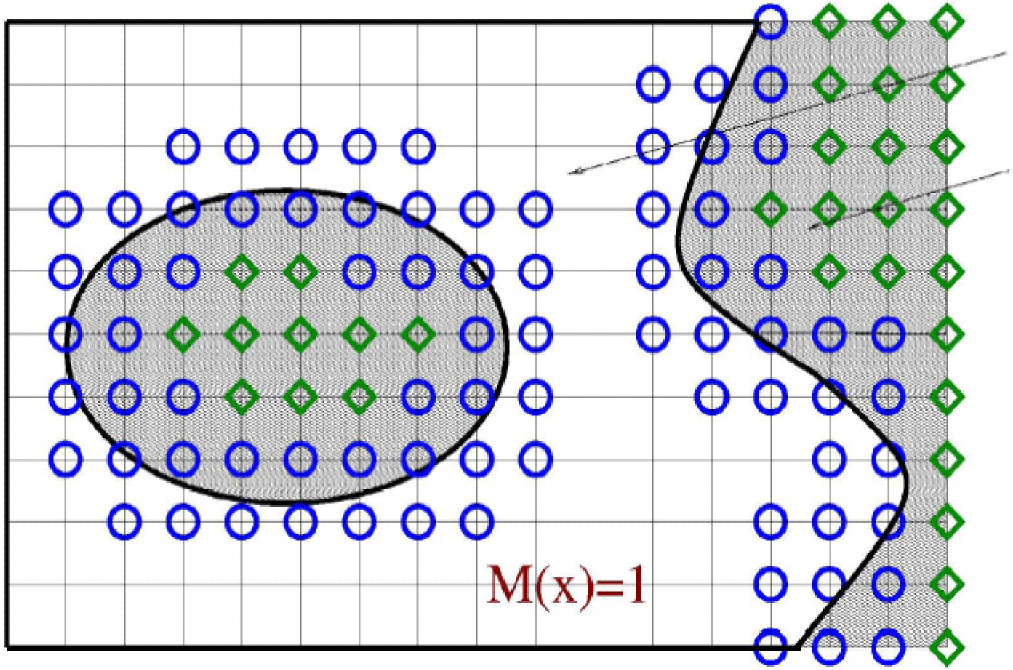
$$\phi^u(\mathbf{x}) = \sum_{\alpha} |\xi^{\alpha}| N^{\alpha}(\mathbf{x}) - \left| \sum_{\alpha} \xi^{\alpha} N^{\alpha}(\mathbf{x}) \right|$$

(Moës, Cloirec, Cartraud, Remacle, cmame 2004)

Note : This function does not require any blending element to reach optimal convergence.



Enrichment for a discontinuity in the matter



Matter

Void

◇ Eliminated nodes

○ Modified function

+ Classical node

(Daux et al. ijnme 1999)

No enrichment, the integration is performed only in the non-void part of the elements (similar to Nitsche).

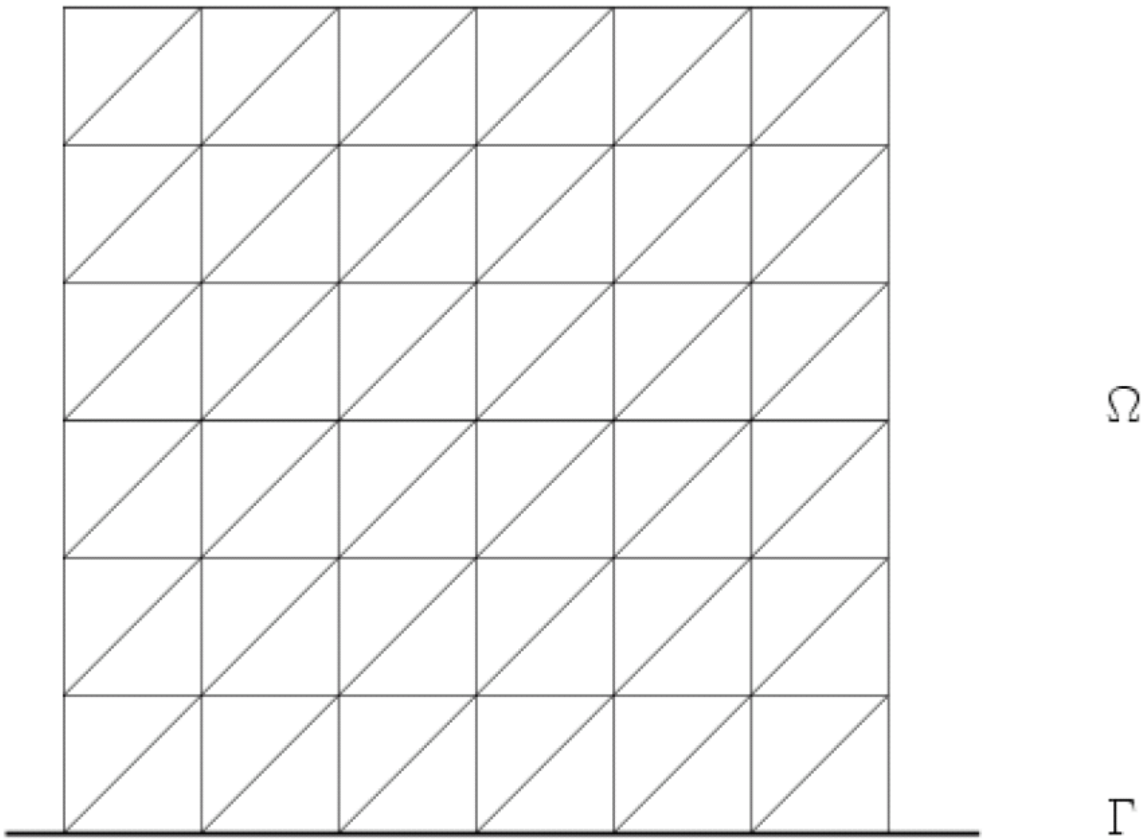
PART 2 :
Stiff Interfaces

- The main issue with Dirichlet or more generally stiff interfaces is to design the correct traction space on the boundary. Once it is designed, any interface model may be handled easily.

Moës, Béchet, Tourbier, IJNME

Géniaut, Massin, Moes, REMN

A picture is worth a …

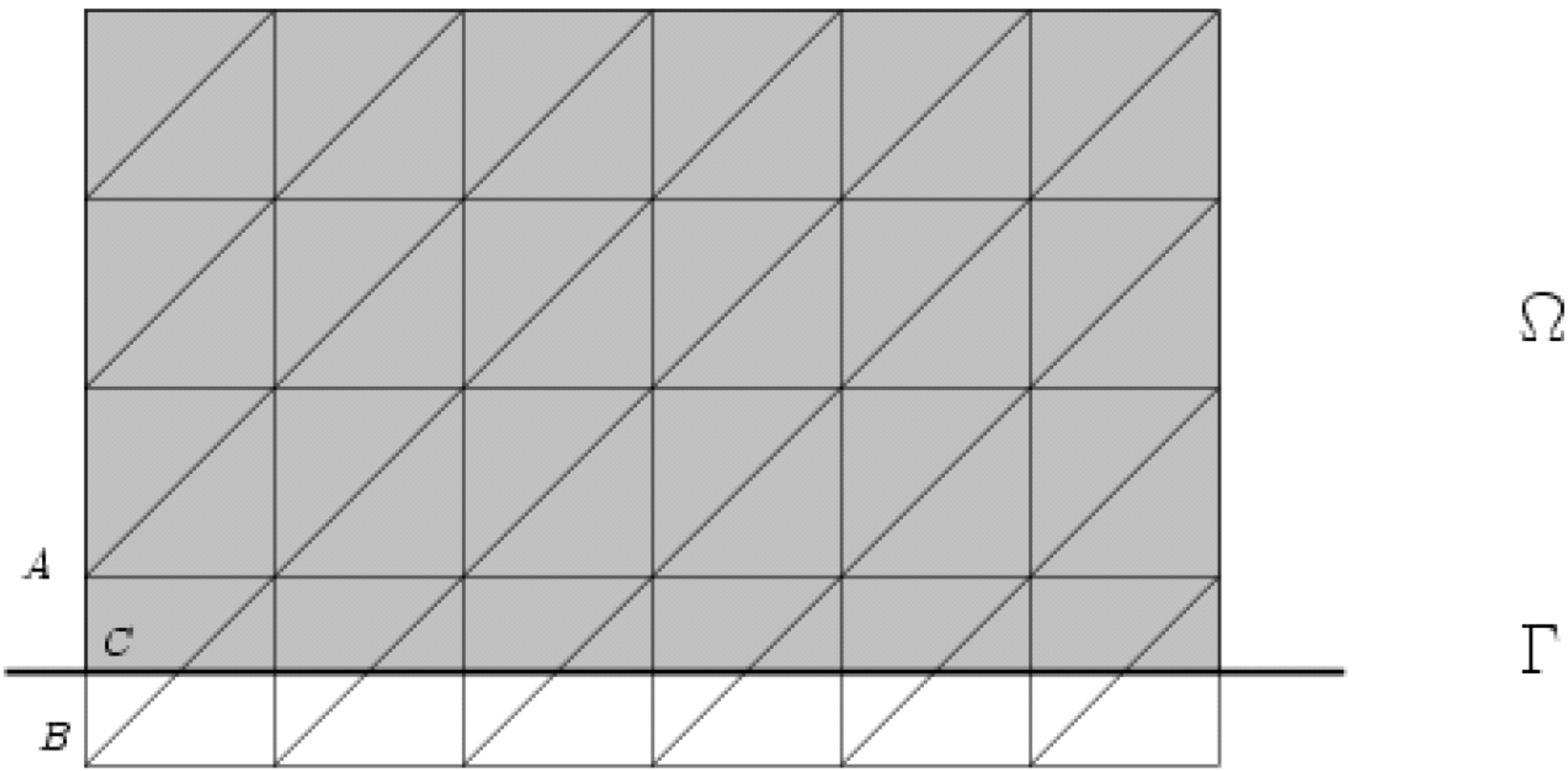


Dirichlet boundary

Matched by the mesh

Size of trace space on Γ : 7

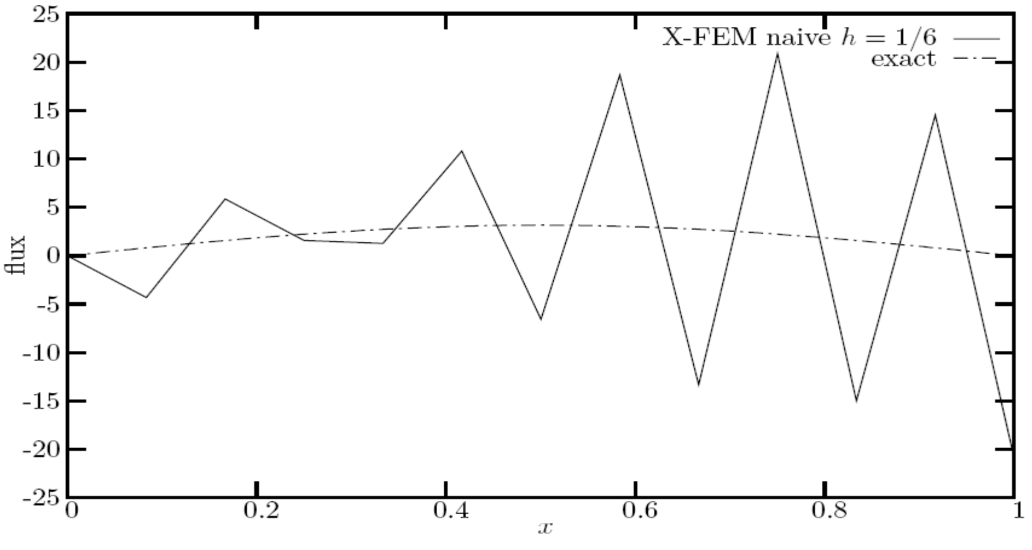
X-FEM case



Size of trace space on Γ : 13

Summary of the Dirichlet issue with X-FEM

- Trace of the inner field on the Dirichlet boundary is very rich when the boundary is not matched.
- If strong imposition of Dirichlet (naive approach), poor fluxes on the boundary (boundary locking) and strong oscillations of the reactions forces (Ji and Dolbow 2004).



We need to relax the Dirichlet imposition

Possible approaches

- **Lagrange multipliers**: Difficulties : choose the correct space. Advantage : reuse of classical contact algorithms.

- Penalty method: where to integrate the penalty term and classical drawbacks (choice of the penalty parameter and degraded conditioning).

- Nitsche’s method: Difficulties : determination of a parameter to insure the stability of the system (inf-sup related in fact). Hard to extend to non-linear bulk behavior. Not easy to reuse all classical contact algorithms.

Lagrange multiplier formulation

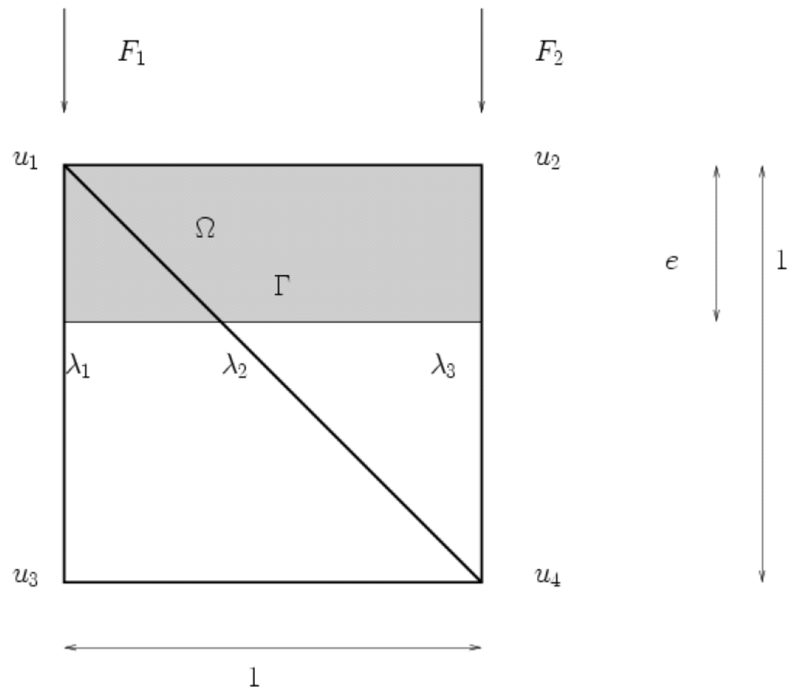
$$
\begin{aligned}
&\text{Find } u \in \mathcal{U}, \lambda \in \mathcal{L}: \\
\int_\Omega \nabla u \cdot \nabla v d\Omega - \int_{\Gamma_\mathrm{D}} \lambda v d\Gamma &= \int_{\Gamma_\mathrm{N}} t_\mathrm{N} v d\Gamma, \qquad \forall v \in \mathcal{U} \\
- \int_{\Gamma_\mathrm{D}} \mu u d\Gamma &= - \int_{\Gamma_\mathrm{D}} \mu u_\mathrm{D} d\Gamma, \qquad \forall \mu \in \mathcal{L}
\end{aligned}
$$

$$
\begin{aligned}
&\text{Find } u^h \in \mathcal{U}^h, \lambda^h \in \mathcal{L}^h: \\
\int_\Omega \nabla u^h \cdot \nabla v d\Omega - \int_{\Gamma_\mathrm{D}}^{\mathrm{num}} \lambda^h v d\Gamma &= \int_{\Gamma_\mathrm{N}} t_\mathrm{N} v d\Gamma, \qquad \forall v \in \mathcal{U}^h \\
- \int_{\Gamma_\mathrm{D}}^{\mathrm{num}} \mu u^h d\Gamma - \int_{\Gamma_\mathrm{D}}^{\mathrm{num}} k^{-1} \lambda^h \mu d\Gamma &= - \int_{\Gamma_\mathrm{D}}^{\mathrm{num}} \mu u_\mathrm{D} d\Gamma, \qquad \forall \mu \in \mathcal{L}^h
\end{aligned}
$$

$$
\begin{pmatrix} A_h & B_h^T \\ B_h & -k^{-1} C_h \end{pmatrix} \begin{pmatrix} U_h \\ L_h \end{pmatrix} = \begin{pmatrix} F_h \\ D_h \end{pmatrix}
$$

Babuska 1973

A deeper look at the naive approach



Case a : λ1 λ2 λ3 indep.

$$\begin{aligned}
\lambda_1 &= F \\
\lambda_2 &= F + \frac{1}{(1-e)}[F] \\
\lambda_3 &= F - \frac{1}{(1-e)^2}[F]
\end{aligned}$$

Case b : λ2 = λ3

$$\begin{aligned}
\lambda_1 &= F - \left(\frac{2e^3 - 9e^2 + 14e - 8}{4e^3 - 12e^2 + 7e + 4}\right)\left(\frac{1}{e}\right)[F] \\
\lambda_2 &= F - \left(\frac{2e^2 - 5e + 4}{4e^3 - 12e^2 + 7e + 4}\right)[F] \\
\lambda_3 &= \lambda_3
\end{aligned}$$

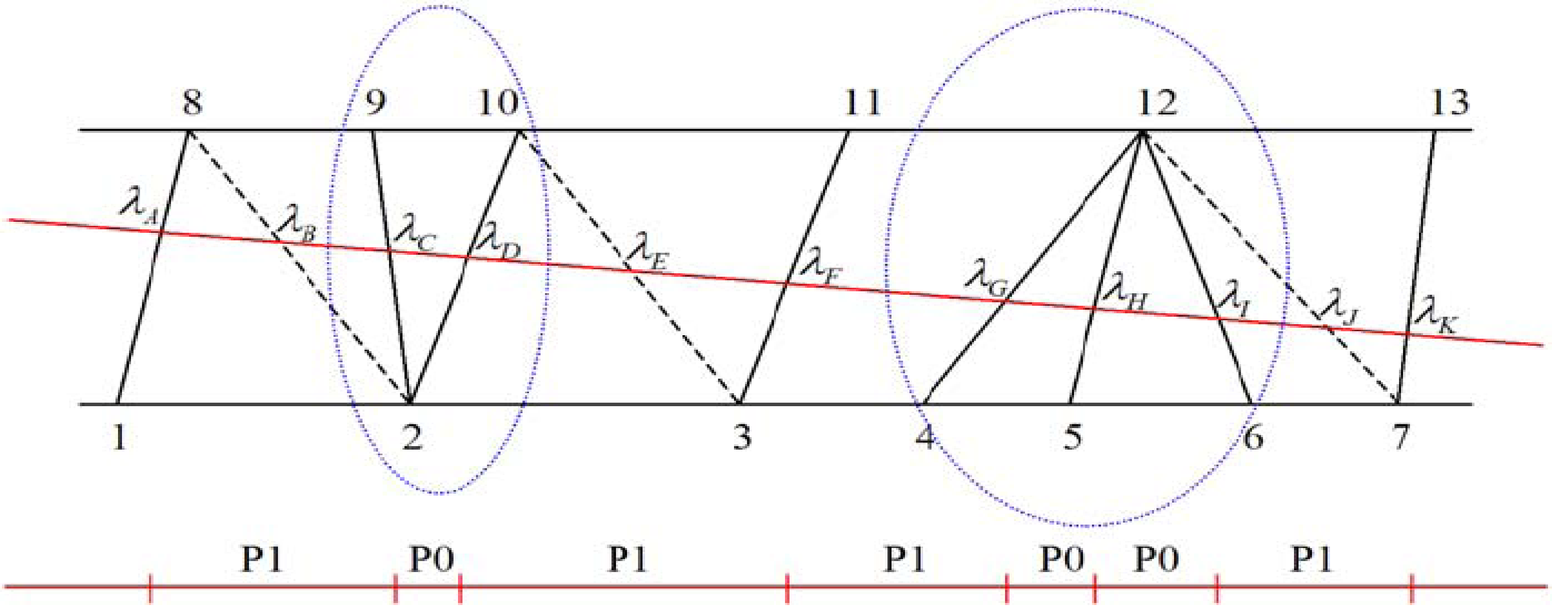
Case c : λ2 = (1-e) λ1 + e λ3

$$\begin{aligned}
\lambda_1 &= F + \left(\frac{2e^3 - 8e^2 + 9e - 4}{4e^5 - 16e^4 + 28e^3 - 32e^2 + 17e - 4}\right)[F] \\
\lambda_2 &= (1-e)\lambda_1 + e\lambda_3 \\
\lambda_3 &= F - \left(\frac{2e^3 - 8e^2 + 9e - 4}{4e^5 - 16e^4 + 28e^3 - 32e^2 + 17e - 4}\right)[F]
\end{aligned}$$

Goal

- Reduction of the Lagrange multiplier space
- Pass the patch test (necessary condition)
- Ensure LBB condition (inf-sup numerical test by Bathe et al.)
- Also check non oscillatory character of the lagrange mutliplier
- Check convergence rates of u et λ

ALGORITHM TO DESIGN THE LAGRANGE MULTIPLIER SPACE (Géniaut et al. REMN).



- Select the essential edges : these are the minimum set of edges connecting the nodes from one side to the other.
- Attach a Lagrange multiplier to each essential edge
- If a set of edges emanate from the same node, they share the same Lagrange multiplier.
- For the non essential edges, the Lagrange multiplier is obtained by linear combination.

Theoretical inf-sup condition

$$\inf_{q^h \in \mathcal{Q}^h} \sup_{v^h \in \mathcal{V}^h} \frac{\int_\Omega q^h \operatorname{div} \mathbf{v}^h \, d\Omega}{\| \mathbf{v}^h \| \| q^h \|} \geqslant \beta > 0$$

β Independant of h

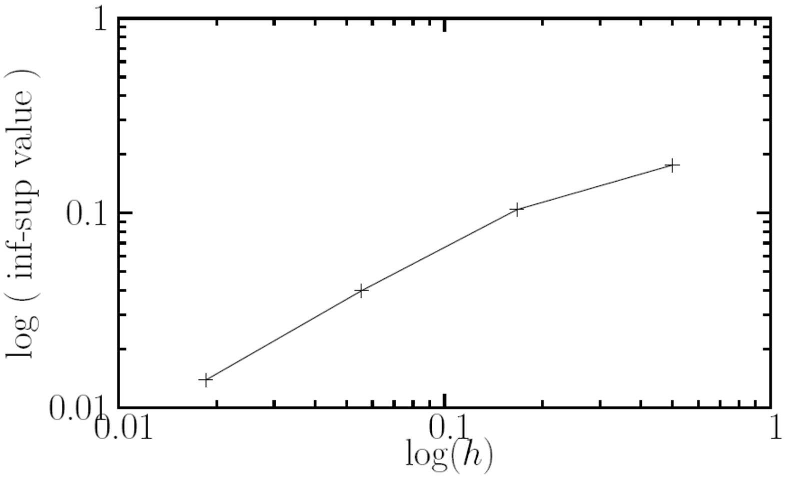
Numerical inf-sup condition (Malkus 81, Chapelle and Bathe 93)

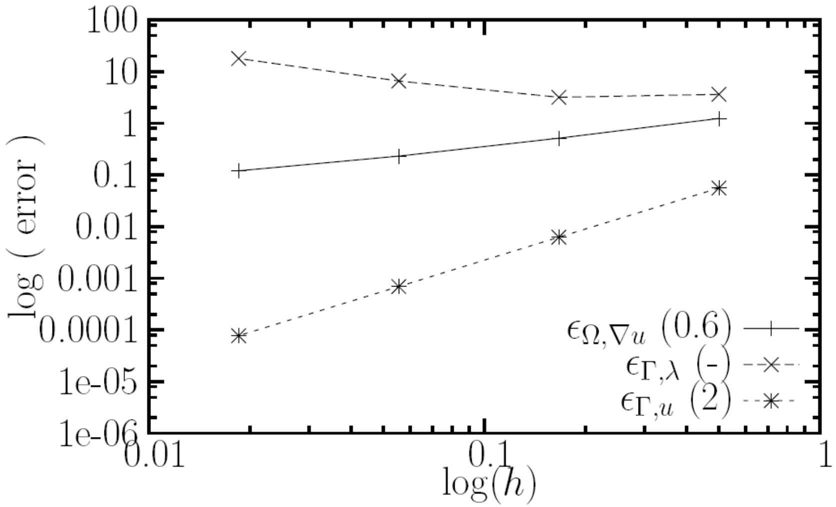
The value of β is simply the min of μ in the following generalized eigen-value problem

$$\mathbf{K_{up}}^T \, \mathbf{M_{uu}}^{-1} \, \mathbf{K_{up}} \, \mathbf{v} = \mu^2 \, \mathbf{M_{pp}} \, \mathbf{v}$$

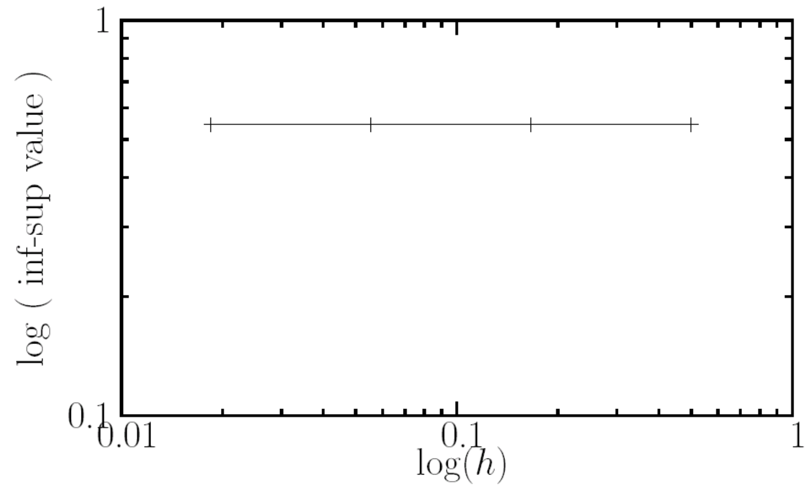
The stability of β with respect to h is checked on a sequence of meshes

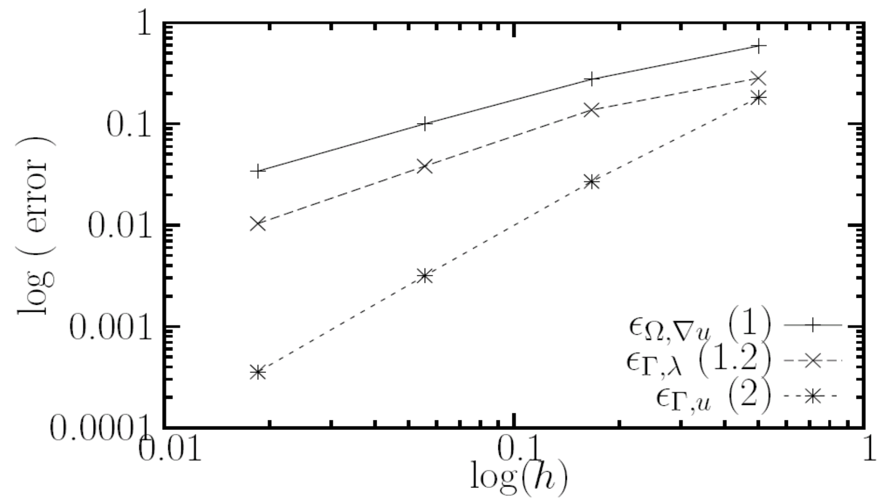
Results with the naive Lagrange multiplier space





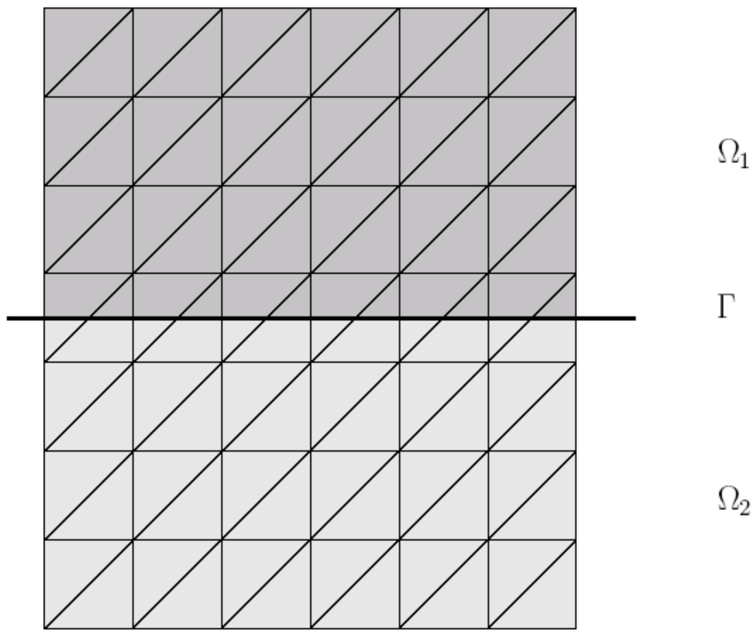
Results with the updated lagrange multplier space





Also, very satisfactory results with various types of meshes

Inner interface condition



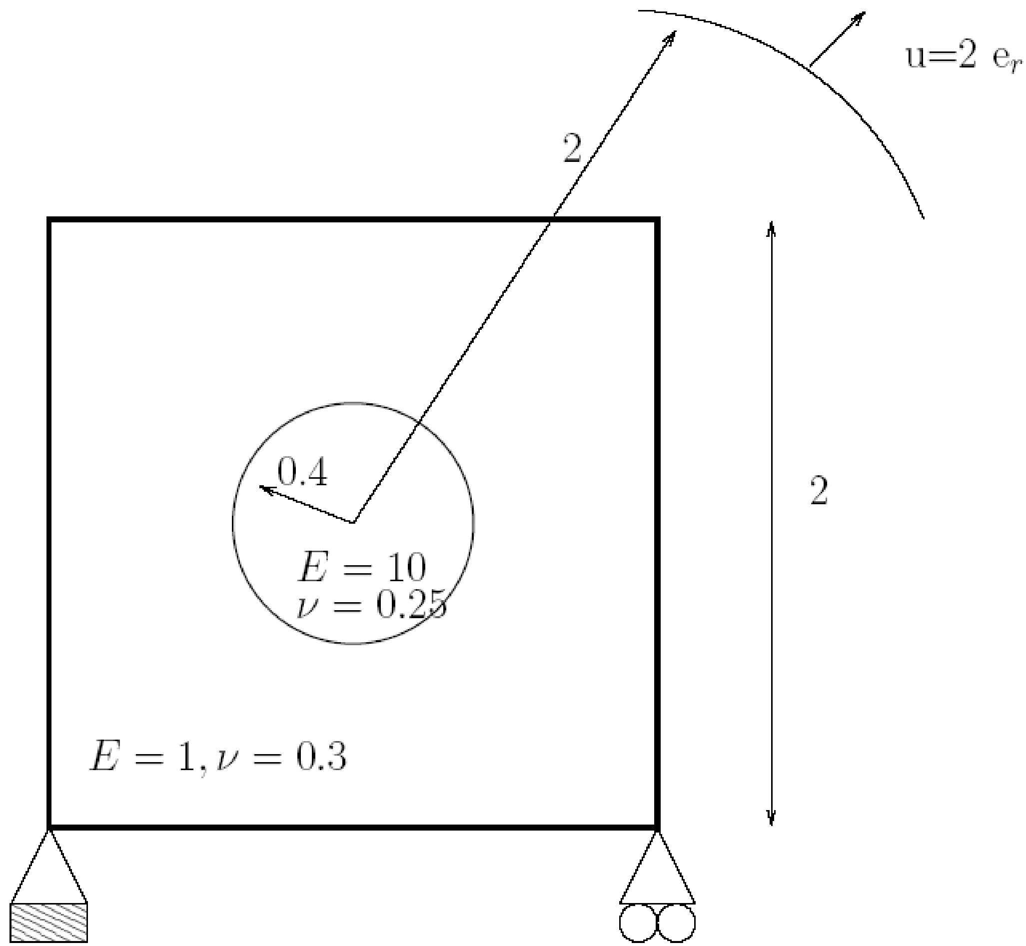
$$u_1 - u_2 = 0, \qquad \frac{u_1 + u_2}{2} = -b(\lambda_1 + \lambda_2) + \overline{u}, \qquad \text{on } \Gamma$$

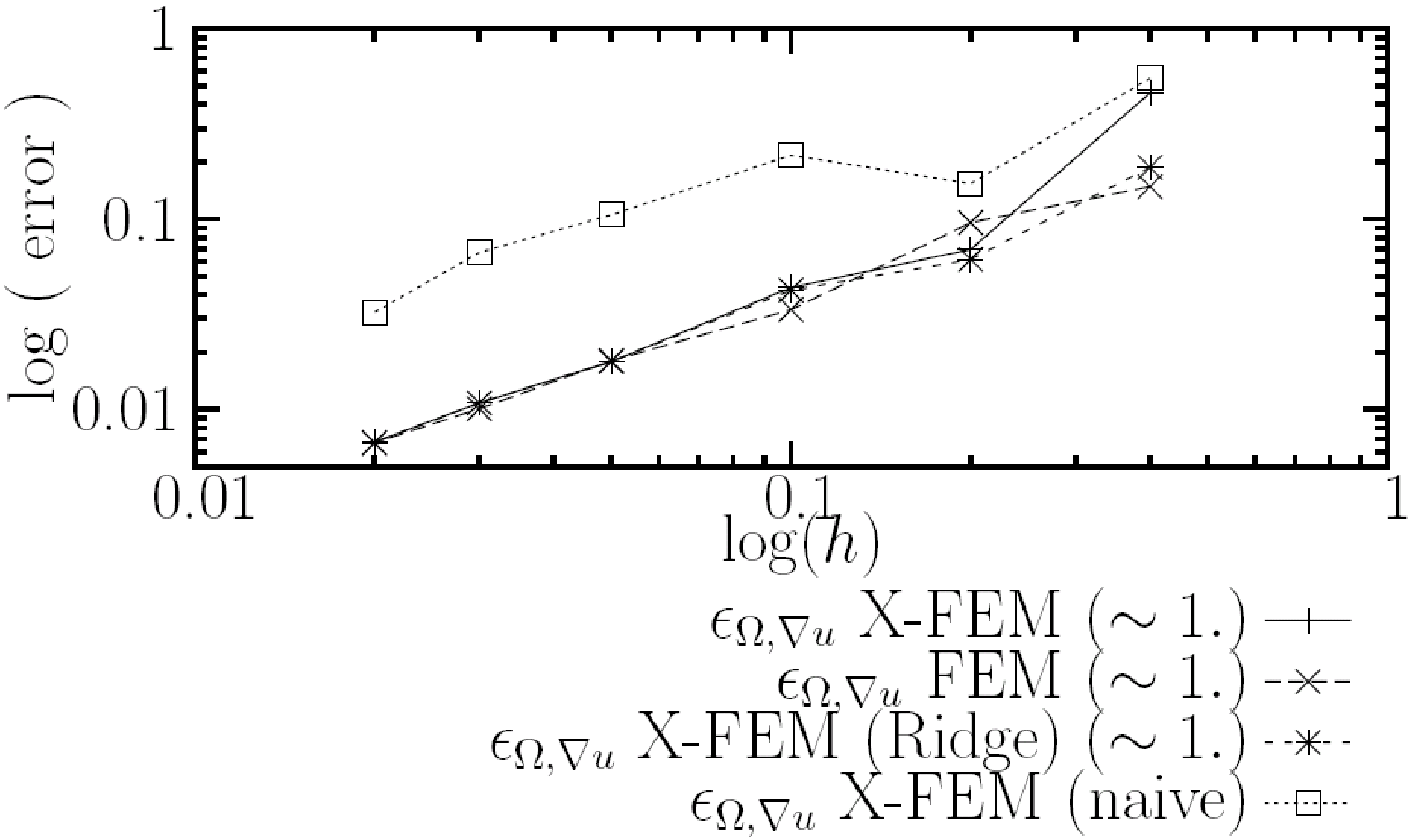
Phase transformation

$$\lambda_1 + \lambda_2 = 0, \qquad \frac{\lambda_1 - \lambda_2}{2} = -k(u_1 - u_2) + \overline{t}, \qquad \text{on } \Gamma$$

Glued material interface

Inner interface condition

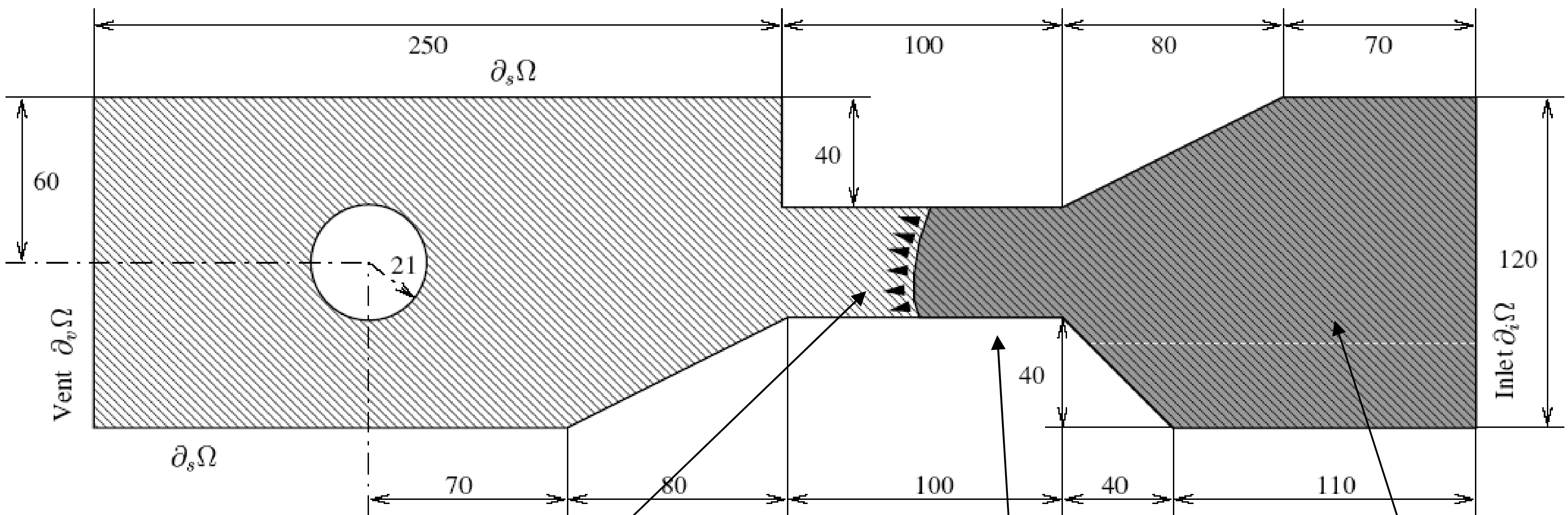




1
0.1
0.01
log (error)
0.01
0.1
1
log(h)
$\epsilon_{\Omega,\nabla u}$ X-FEM ($\sim$ 1.)
$\epsilon_{\Omega,\nabla u}$ FEM ($\sim$ 1.)
$\epsilon_{\Omega,\nabla u}$ X-FEM (Ridge) ($\sim$ 1.)
$\epsilon_{\Omega,\nabla u}$ X-FEM (naive)

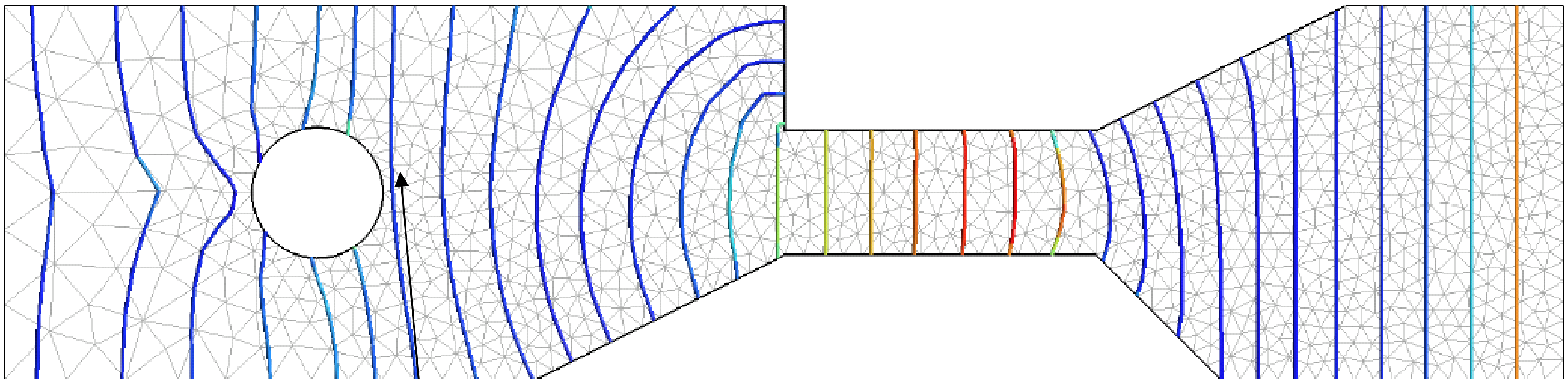
PART 3 Applications

Application to an RTM mold filling problem



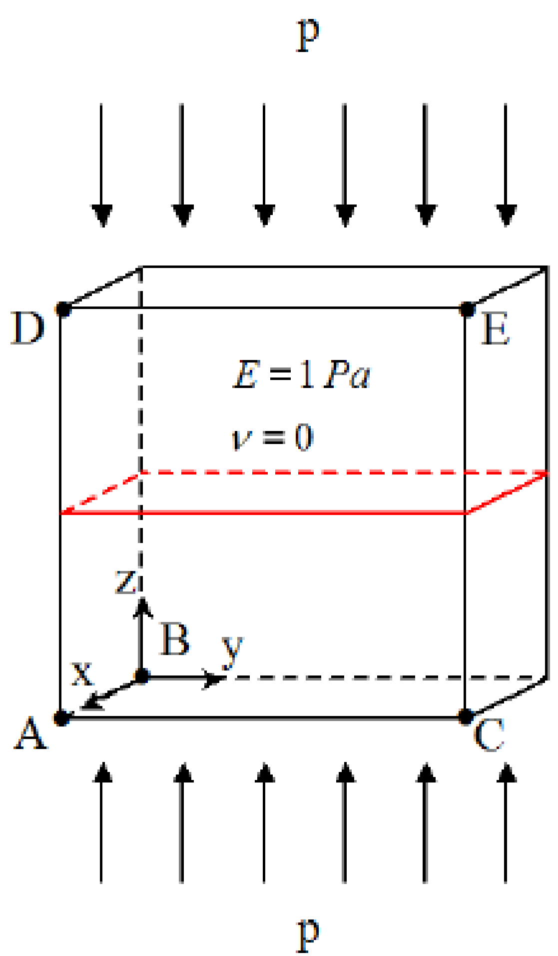
$$p = 0 \qquad \nabla p \cdot \mathbf{n} = 0 \qquad \Delta p = 0$$

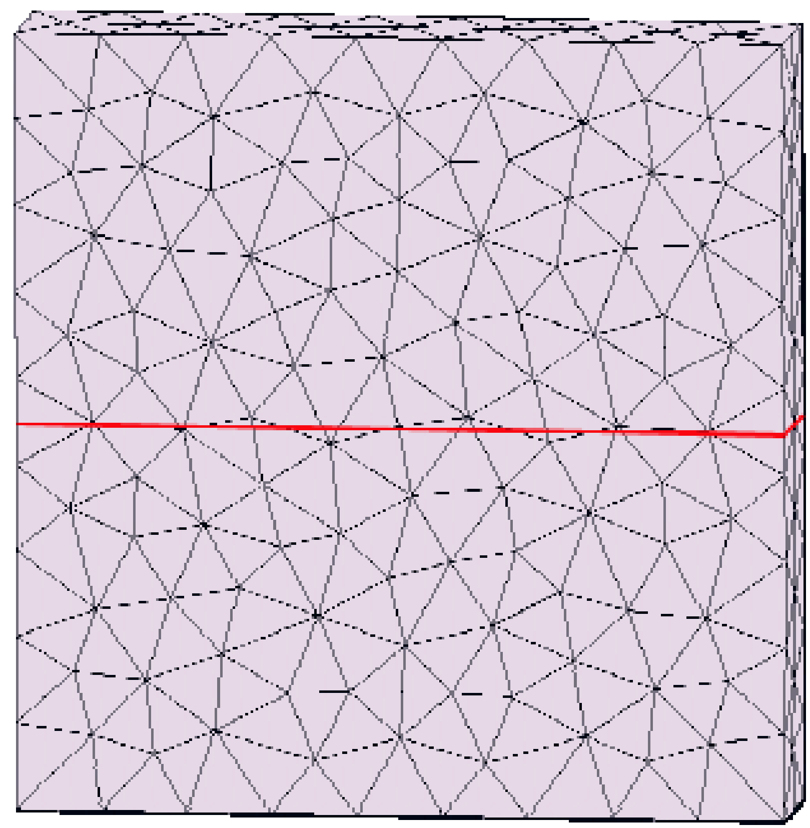
Evolution of the front position with time



Topological change of the front treated by level sets

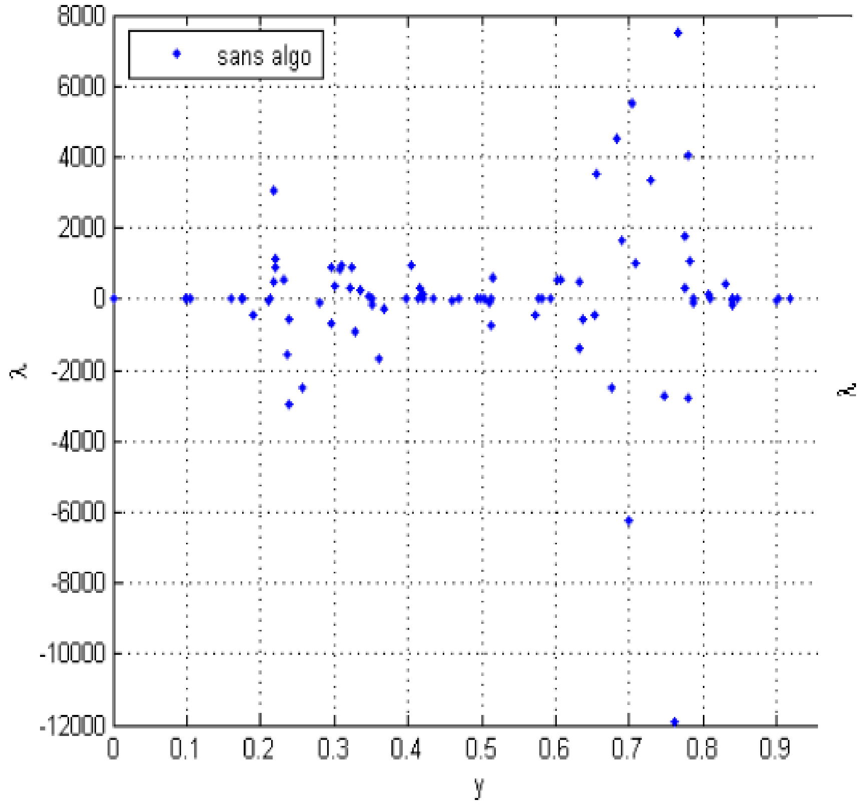
Contact patch test (no friction)

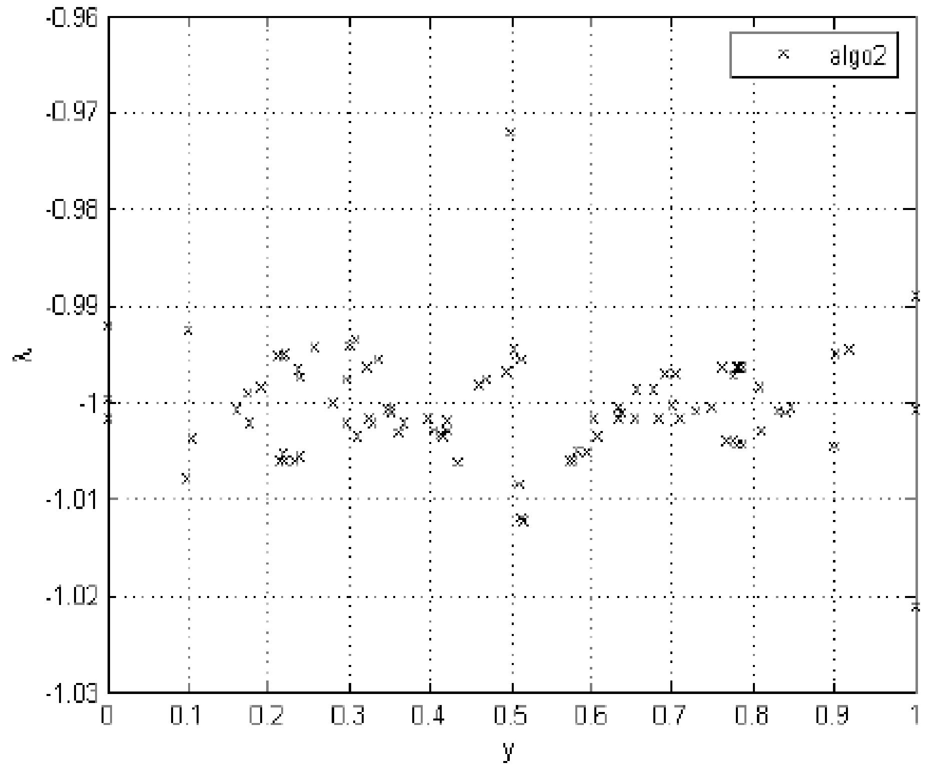




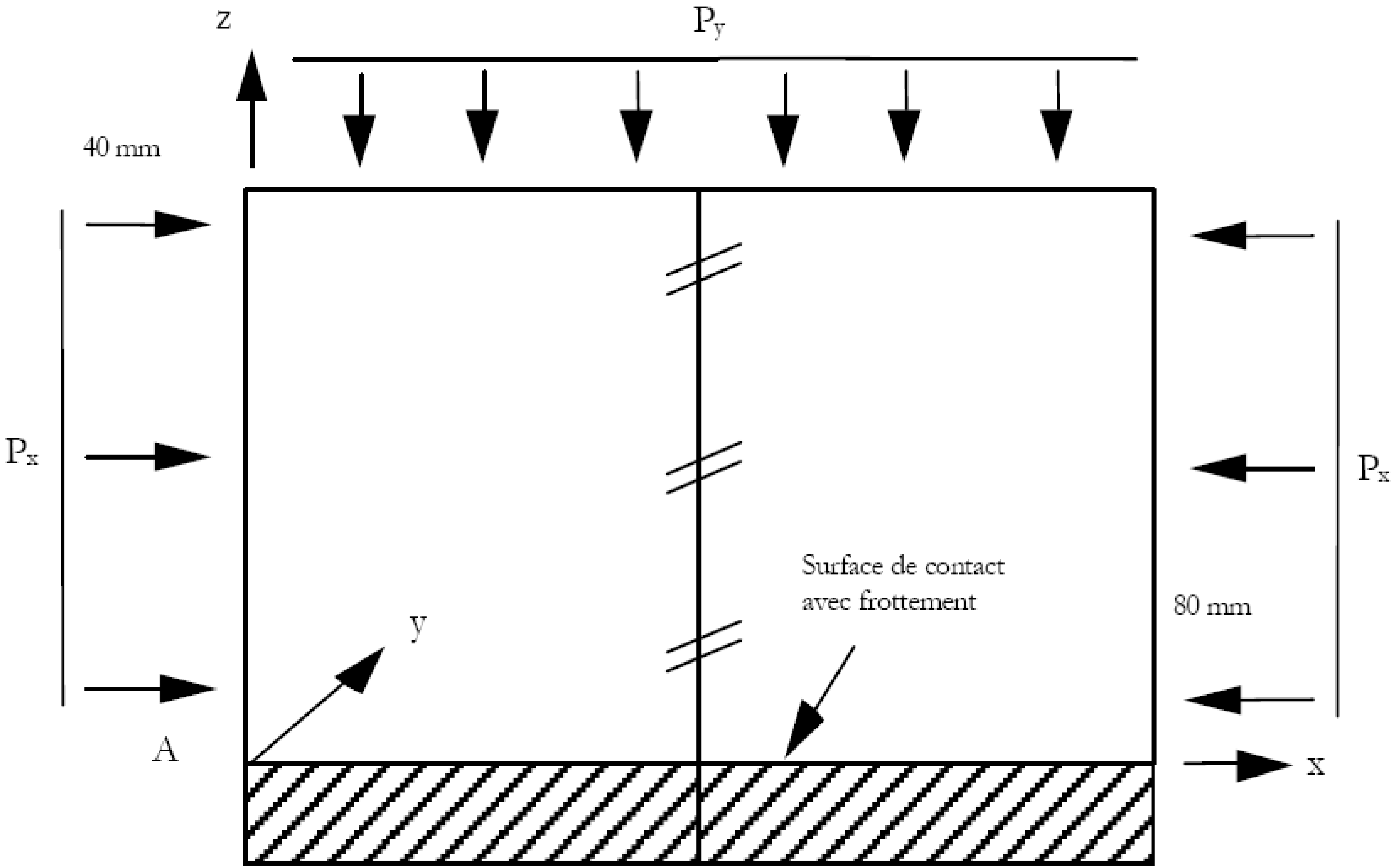
Geniaut PhD thesis

Results for the pressure with the naive and reduced space

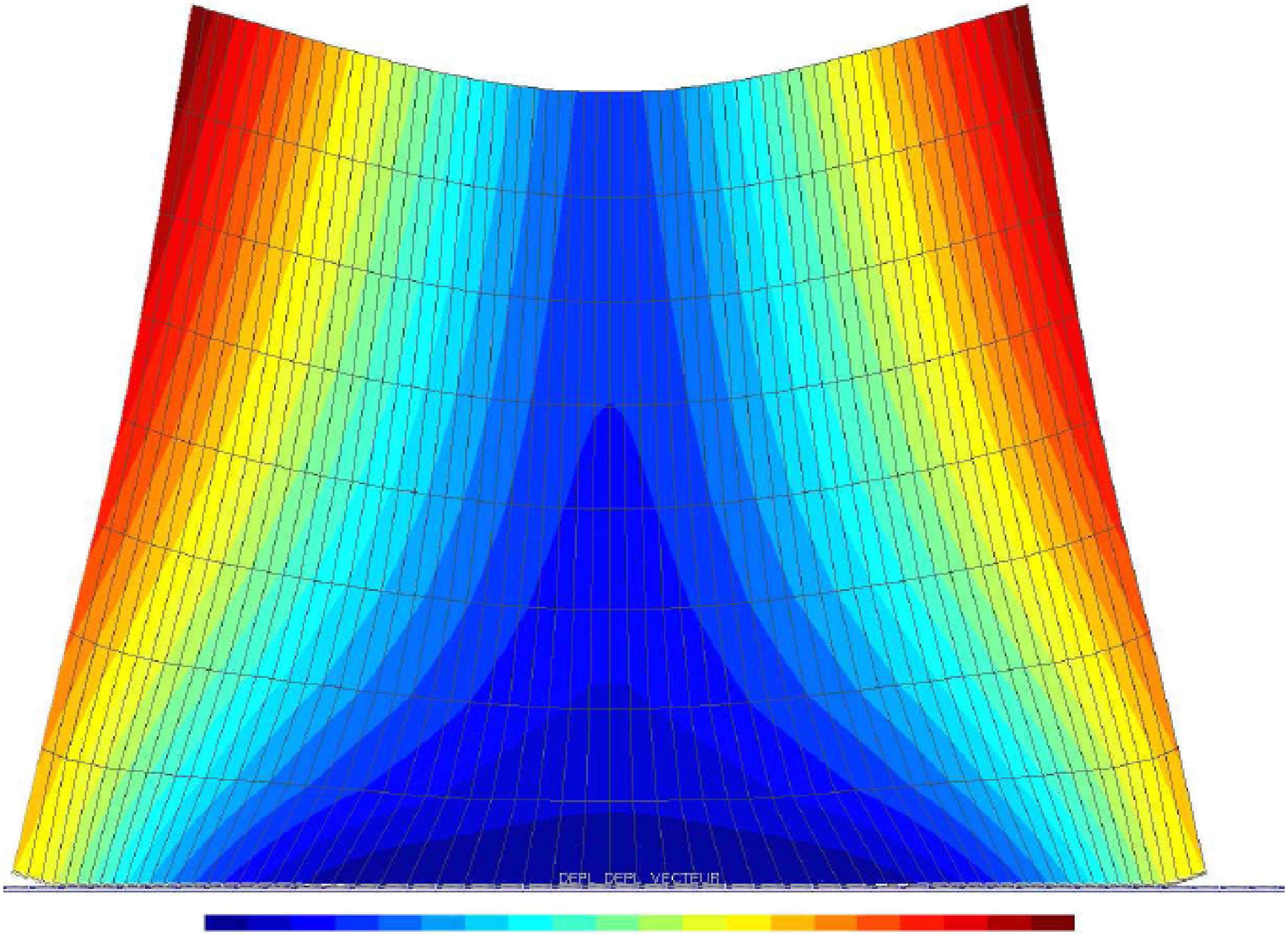




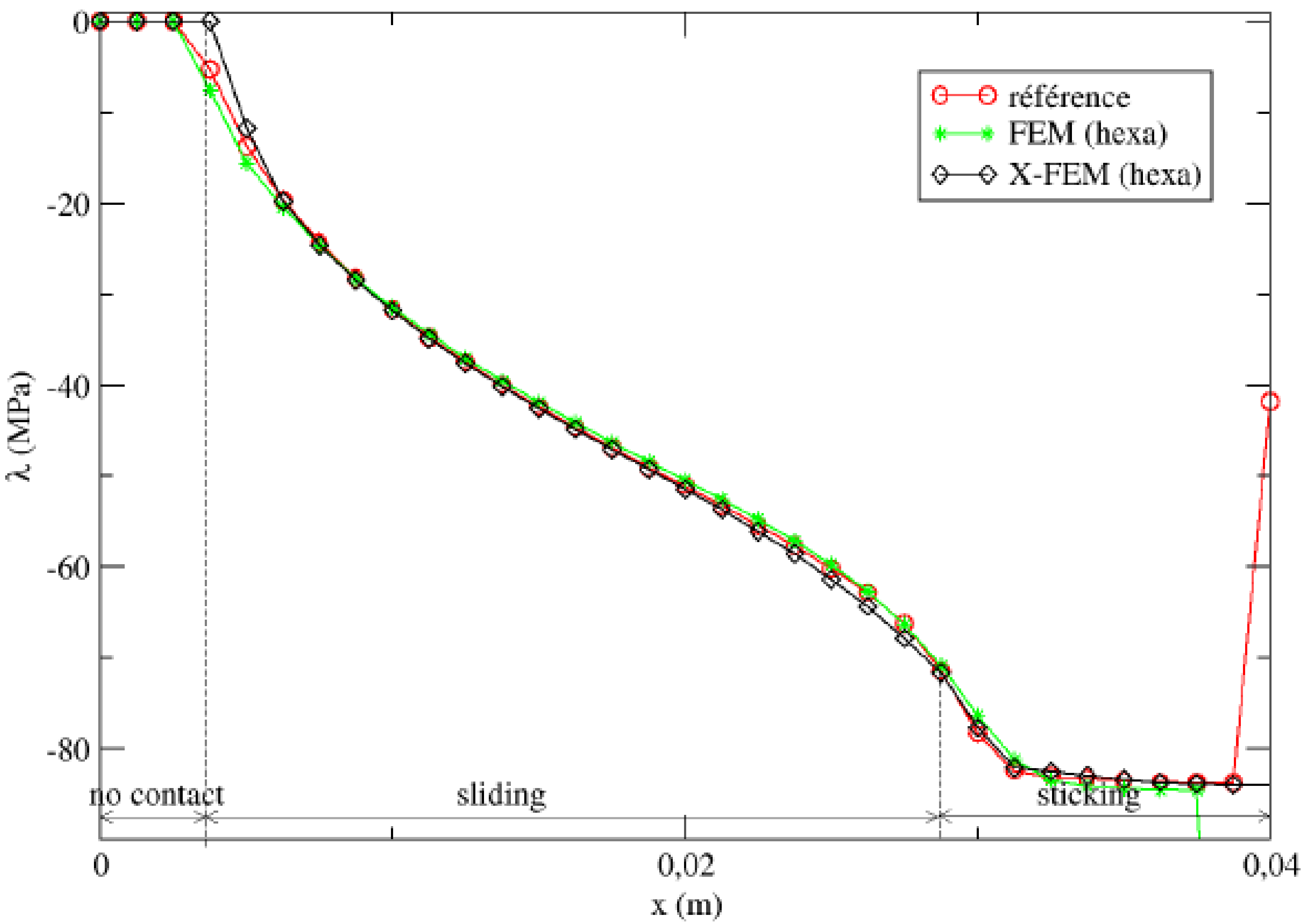
Contact with friction (Geniaut thesis)



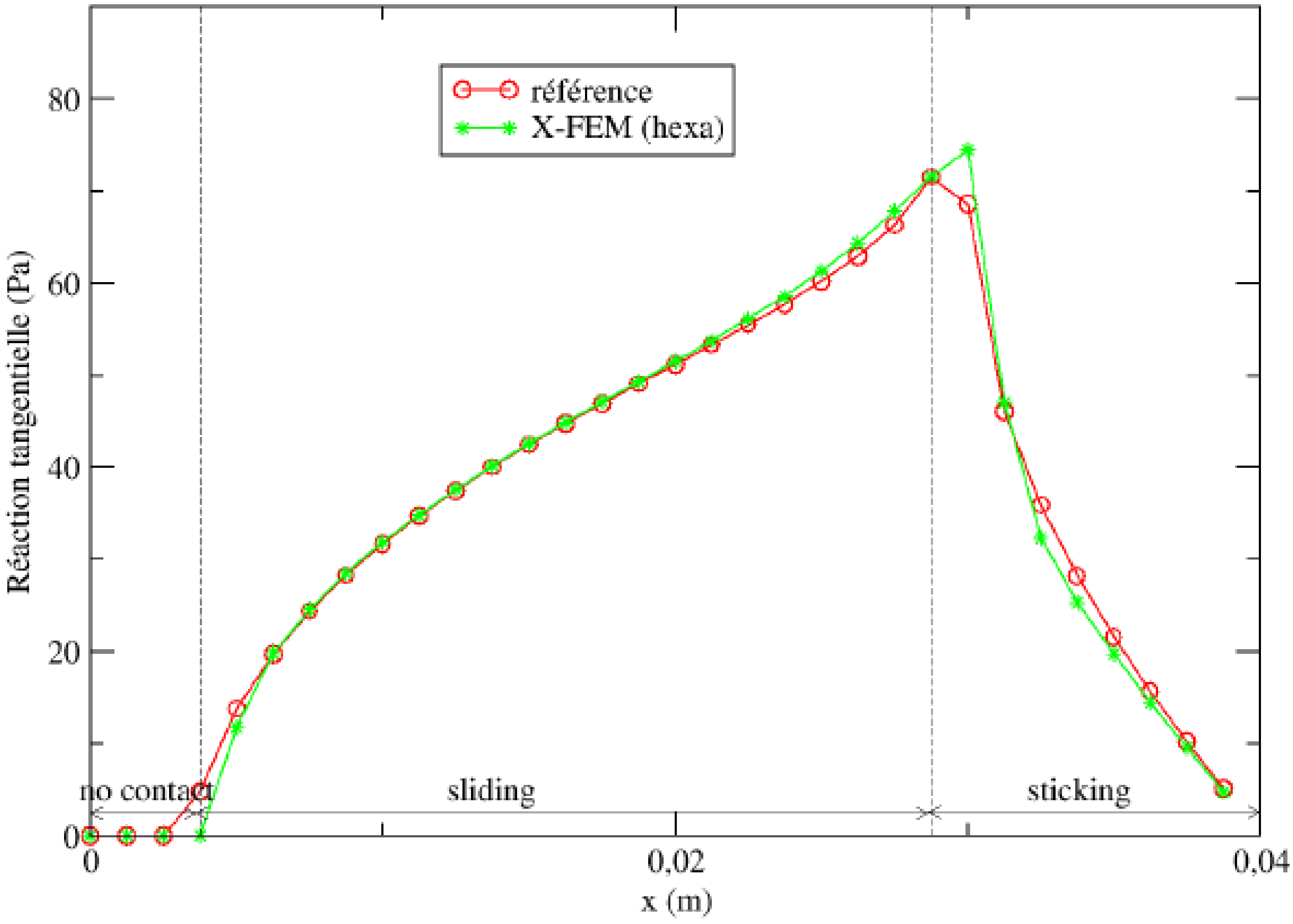
Deformed shape (exagerated)



Contact pressure



Tangential force



CONCLUSIONS

•Great care must be taken to impose stiff boundary conditions in X-FEM (as in FEM...).

• The naive approach fails and the reason is the exceeding size of the Lagrange multiplier space

•A systematic approach to reduce the space is proposed and removes the locking

•The approach seems applicable to all types of stiff constitutive law on the interface.

•Contact with friction was taken into account (with Ben Dhia continuous contact formulation) as well as mold filling with a potential fluid.