Particles and interfaces

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Outline:

-Remeshed Particle Methods and High Order Finite Differences
-Two-way interaction of rigid solid and incompressible flow

Rule of the game:

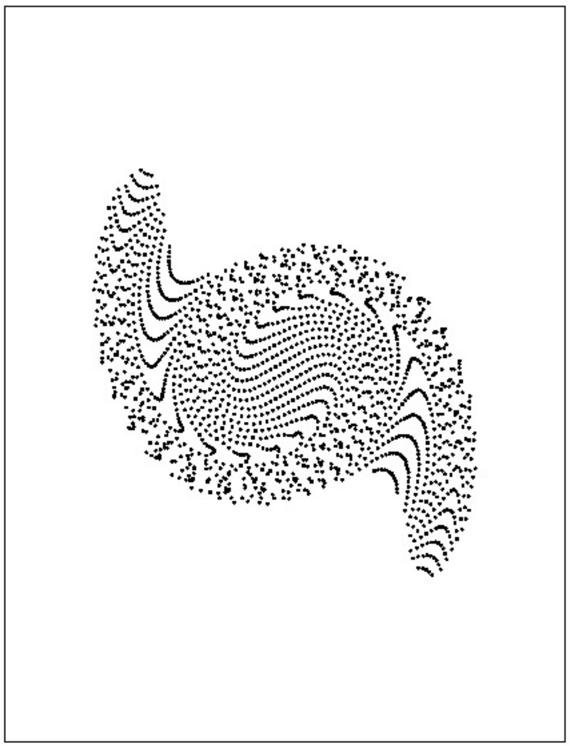
$$\frac{\partial U}{\partial t} + \operatorname{div}\left[G(U)U\right] = F(U)$$

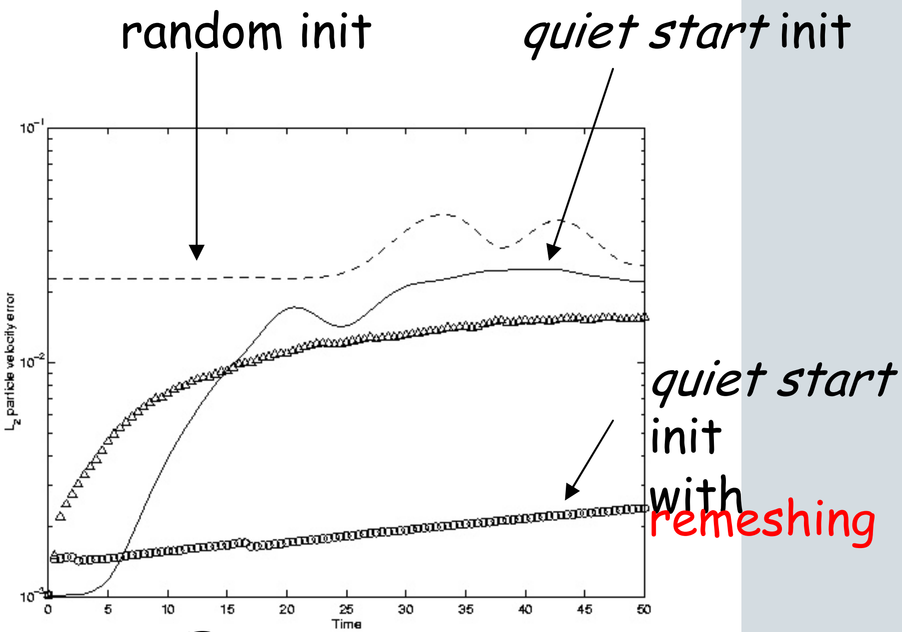
$$U(x) \approx \sum_p v_p U_p \delta(x - x_p)$$

$$\frac{dx_p}{dt} = G(U_p) \ ; \ \frac{dU_p}{dt} = F(U_p) \ ; \ \frac{dv_p}{dt} = (\operatorname{div} G)(U_p)$$

Particle methods:

Typical and simple example showing importance of regridding to maintain acuracy: circular patch with high strain





Error curves

For this particular example other techniques can avoid remeshing but for 3D flows, remeshing cannot be avoided to obtain reliable results

Remeshing can also be used to

- Take advantage of underlying grid to compute non advection-related terms (fields, pressure gradients, surface tension ..)
- Provide natural coupling techniques in particle/particle or particle/grid domain decomposition methods
- Impose variable resolution where needed (multilevel particle methods)

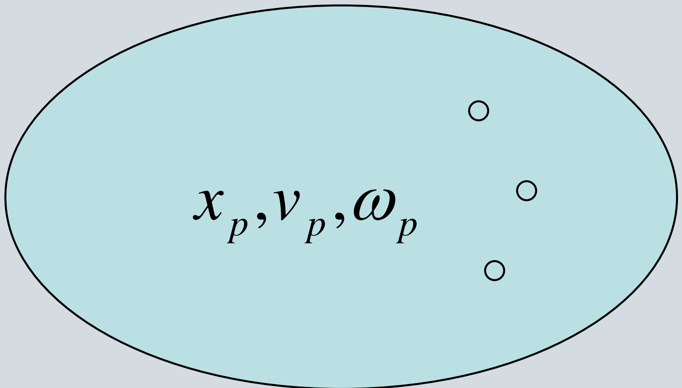
Question:
what remains of the Lagrangian features of a particle method after remeshing?

Regridding can be used as a way to adapt particles to the flow topology -> variable-size particles

three different approaches:

- Regridding into variable-sized particles via global mappings
- Regridding via local mappings
- Regridding onto piecewise uniform particles

Physical space

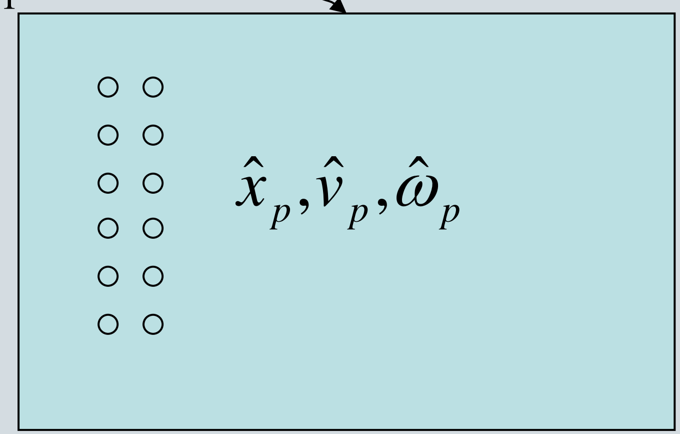


$$x = F(\hat{x}); G = F^{-1}$$

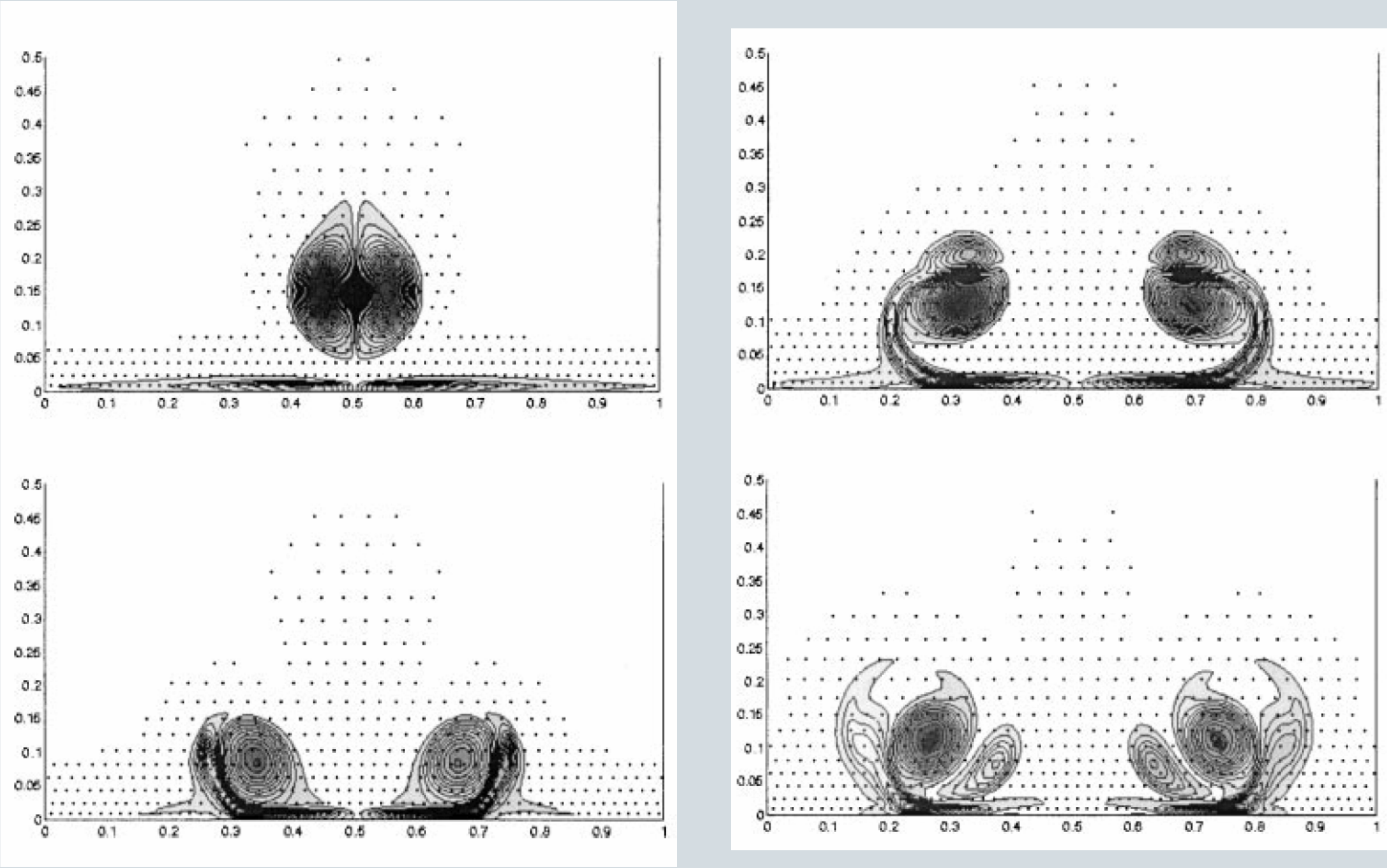
$$a_{ij} = \frac{\partial G_i}{\partial x_j}$$

$$J = \det[a_{ij}]$$

$$v_p J(\mathbf{x}_p) = \hat{v}_p$$



Mapped space



Example: rebound of a dipole with exponentially stretched particle distribution
(C-Koumoutsakos-OuldSalihi, JCP 00)

Variable-size particles can be used in an adaptive fashion

Two possible strategies (Bergdorf,C.,Koumoutsakos, SIAM MMS):

•Adaptively build global mappings

•Adaptively define zones with piecewise constant volumes (AMR approach)

Second approach: particles have piecewise constant volumes

Inside each population, particles dynamics is straightforward
Remeshing allows to transfer information between populations through a buffer zone:

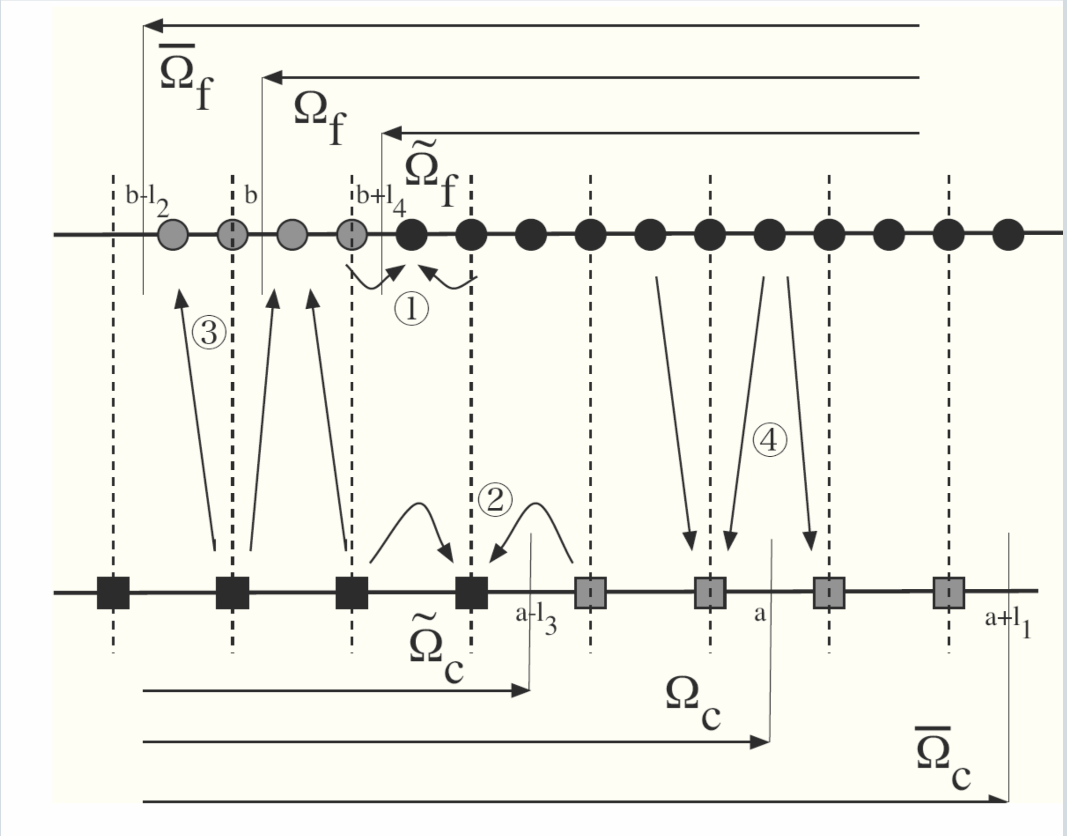
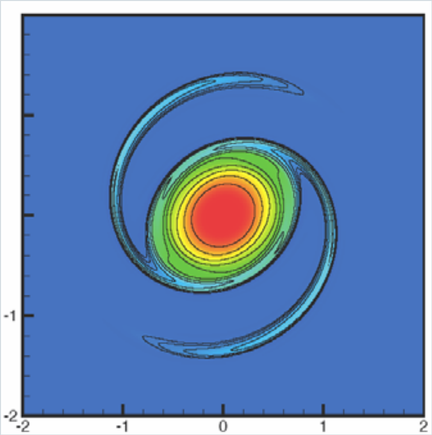
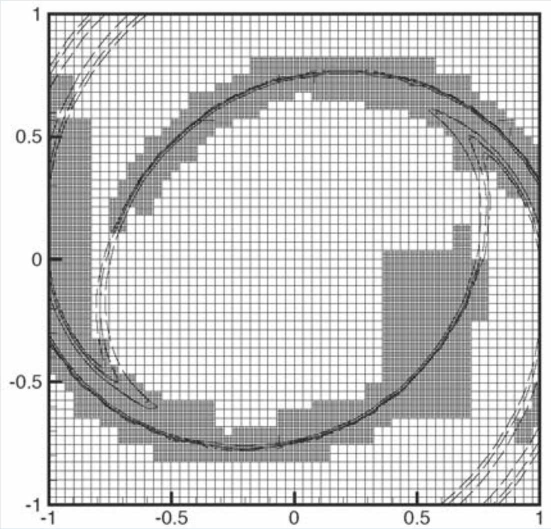


Illustration: case of an elliptical vortex



Adaptation based on vorticity gradients
Fine and coarse zone in the AMR approach
defined using a package designed for FD

AMR approach:



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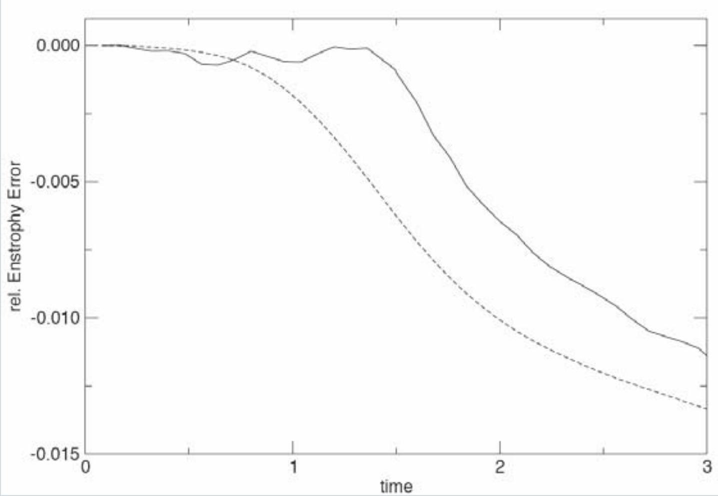
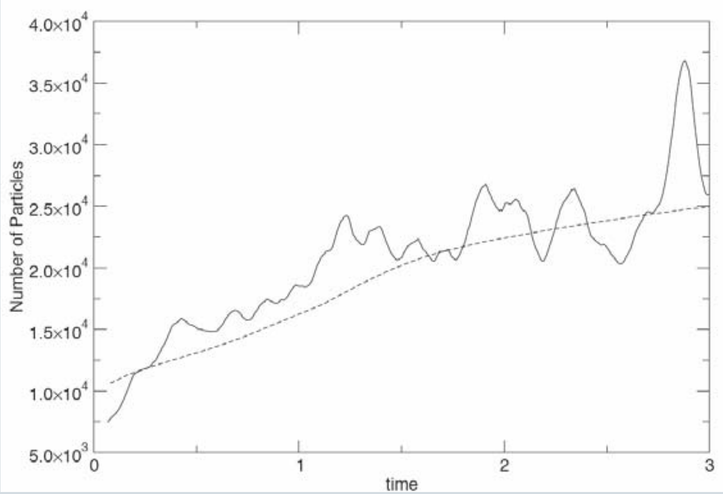


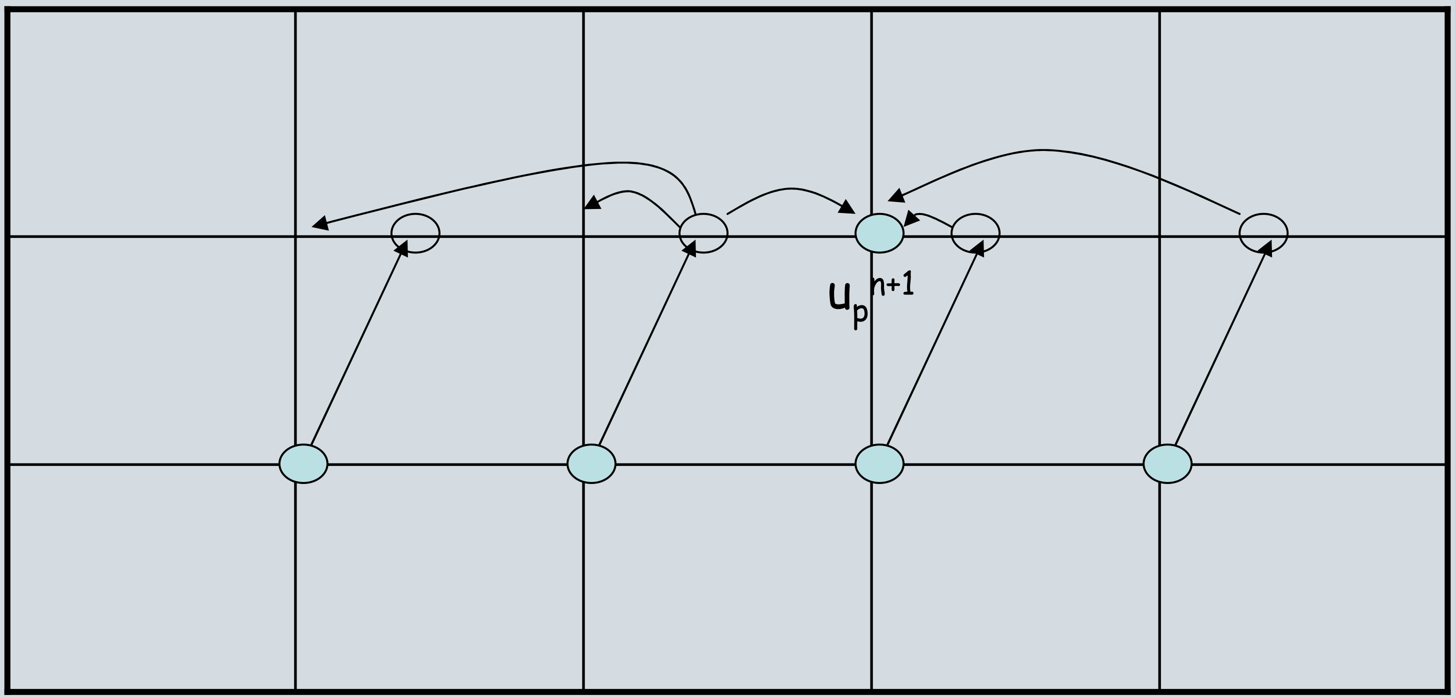


Block definition strategy can be simplified using wavelet-based multiresolution flow analysis (Bergdorf-Koumoutsakos)

Remeshing particles at each time-step deserves direct FD-type analysis

Toy problem: $u_t + a u_x = 0 \ , \ a > 0$

Focus on remeshing formula conserving 3 first moments (Λ_2 formula):



If $\Delta t < a\Delta x/2$: $u_p^{n+1} = c_0 u_p^n + c_{+1} u_{p+1}^n + c_{-1} u_{p-1}^n$

Conservation of 3 first moments gives

$$c_{-1} + c_0 + c_{+1} = 1\ ,\ c_{-1} - c_{+1} = \lambda\ ,\ c_{-1} + c_{+1} = \lambda^2$$

Where $\lambda = a\Delta t/\Delta x$

$$c_0 = 1 - \lambda^2, c_{\pm 1} = \mp\lambda(1 \mp \lambda)/2$$

This is Lax-Wendroff
If $1 > \lambda > 1/2$, upwind version of Lax-Wendroff
If $\lambda > 1$, combination of L-W and exact solution

Higher order conservation gives finite-difference method of corresponding order (C., Weynans, CRAS 2006)

Case of non-linear problems: $u_t + (g(u)u)_x = 0$

Need some care to define second order particle time-stepper

Idea: evaluate particle velocities at time $t_n+\Delta t/2$ using following equation

$$\frac{du}{dt} + (g(u))_x u = 0$$

Particle method goes along these lines:
First compute

$$u_p^{n+1/2} = u_p^n\left(1 - \Delta t g(u^n)_x(x_p)/2\right)$$

Where $g(u^n)_x(x_p)$ computed by centered f.d. on the grid

then advance particles from t_n to t_{n+1} with velocity $\tilde{g}_p^n = g(u_p^{n+1/2})$

Finite-difference interpretation:

$$u_p^{n+1} = u_p^n - \frac{\Delta t}{2h}(\tilde{g}_{p+1}^n u_{p+1}^n - \tilde{g}_{p-1}^n u_{p-1}^n) +$$

$$\frac{\Delta t^2}{2h^2}\left((\tilde{g}_{p+1}^n)^2 u_{p+1}^n - 2(\tilde{g}_p^n)^2 u_p^n + (\tilde{g}_{p-1}^n)^2 u_{p-1}^n\right)$$

Can prove that scheme is second order and equivalent equation given by

$$u_t + (g(u)u)_x +$$

$$h^2\left[\frac{1}{6}\left(g(u)u\right)_{xxx} + \frac{\mu^2}{6}u_{ttt} + + \frac{\mu^2}{8}(g(u)_x^2 u^3 g''(u))_x + \frac{\mu^2}{2}\left(g(u)g'(u)u^2 g(u)_x\right)_{xx}\right] = 0$$

More: can prove that if supplemented by appropriate von-neumann artificial viscosity, particle scheme is energy decreasing

Remarks:

- Same analysis applies to other remeshing formulas with expected truncation error
- Because of fd needed to evaluate velocities at $t_{n+1/2}$, seems to be a 5-points scheme

In fact: in *d* dimensions, with field G(u), need to evaluate the divergence of G:
Cost is $O(2d+3^d) \ll O(5^d)$

Even better: for hydro codes: U= $(\rho U)/\rho$ and, to leading order,

$$U_p^{n+1/2} = \frac{(\rho U)_p^n \left(1 - \Delta t \,\mathrm{div} G_p^n /2\right)}{\rho_p^n \left(1 - \Delta t \,\mathrm{div} G_p^n /2\right)} = \frac{(\rho U)_p^n}{\rho_p^n}$$

(Euler scheme)

Analysis does not reveal specific features of particle methods:

- Convergence to entropy solutions
- Spectral profile of sub-grid dissipation

More needs to be done to understand optimal artificial viscosity

Comparison of equivalent equations for Lax-Wendroff and remeshed particle methods with Λ_2 and Λ_3 remeshing:
Energy dissipation predicted for Burger equation ($g(u)=u/2$, $\mu = \Delta t/\Delta x$)

Lax-Wendroff:

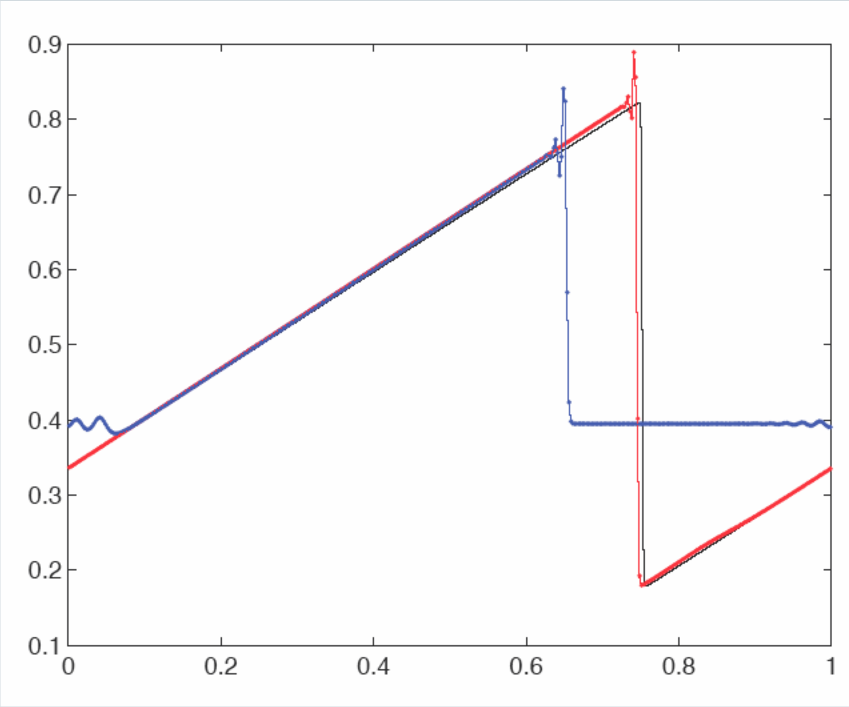
$$\frac{d}{dt}\|u\|^2 = \Delta x^2 \int \left(\frac{1}{2} - \frac{1}{4}|\mu u|^2\right) u_x^3 \, dx$$

Particle method with Λ_2 remeshing :

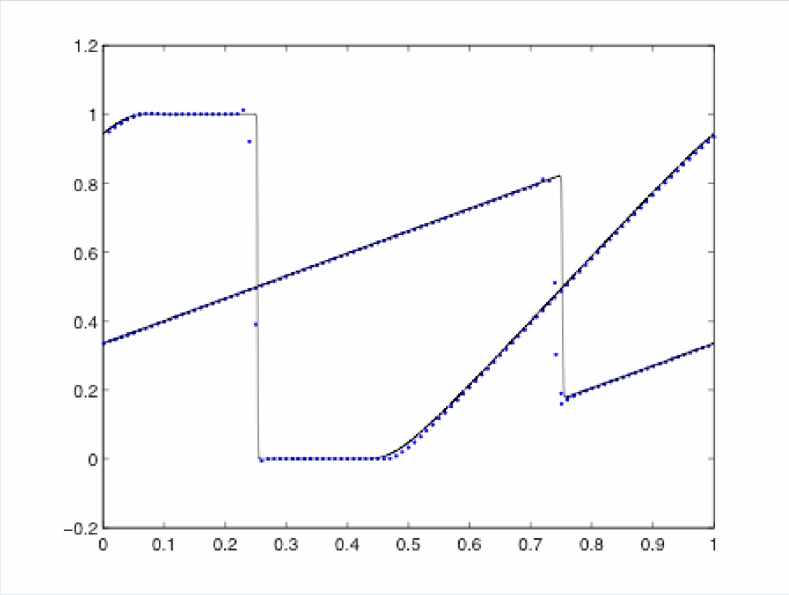
$$\frac{d}{dt}\|u\|^2 = \Delta x^2 \int \left(\frac{1}{2} - \frac{1}{16}|\mu u|^2\right) u_x^3 \, dx$$

However schemes behavior notably different: LW is not entropic, PM is ..

Burgers equation with IC = top-hat function



Blue=LW, red=PM with Λ_2, black=exact



PM with Λ_2 and artificial viscosity at 2 successive times

Example of particle method for interface problem: fluid interacting with rigid body

(with Coquerelle & Cani, 2006)

Two approaches:

1) solve separately fluid and solid and impose continuity
2) Consider system as single flow with variable density and Constitutive laws

Second approach more numerically efficient

Idea: formulate problem as a penalization model :

$$\boxed{\rho\left(\frac{\partial u}{\partial t}+\operatorname{div}(u:u)\right)+\nabla p=\mu\Delta u+\rho \mathrm{g}+\lambda\chi_S(\bar{u}-u)}$$

where $\bar{u}$ is the rigid displacement obtained by averaging velocities over S and $\lambda >> 1$.

Curl of the equation gives:

$$\boxed{\frac{\partial \omega}{\partial t}+\operatorname{div}(u\omega)-(\omega.\nabla)u=\mu\Delta\omega+\nabla\times(\rho g)+\lambda\chi_S(\bar{\varpi}-\omega)+\lambda\delta_{\partial S}n\times(\bar{u}-u)}$$

$$\boxed{\frac{\partial \chi}{\partial t}+\operatorname{div}(\chi u)=0\ ,\ \rho=\rho(\chi)\ ,\ \bar{u}=\frac{\int u\chi\,dx}{\int \chi\,dx}+\frac{\int (r\times u)\chi\,dx}{\int r^2\chi\,dx}\times r}$$

Two vortex generators on the interface.
Surface term δ, ρ and χ recovered through a renormalized level set method.

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Tumbling of spheres

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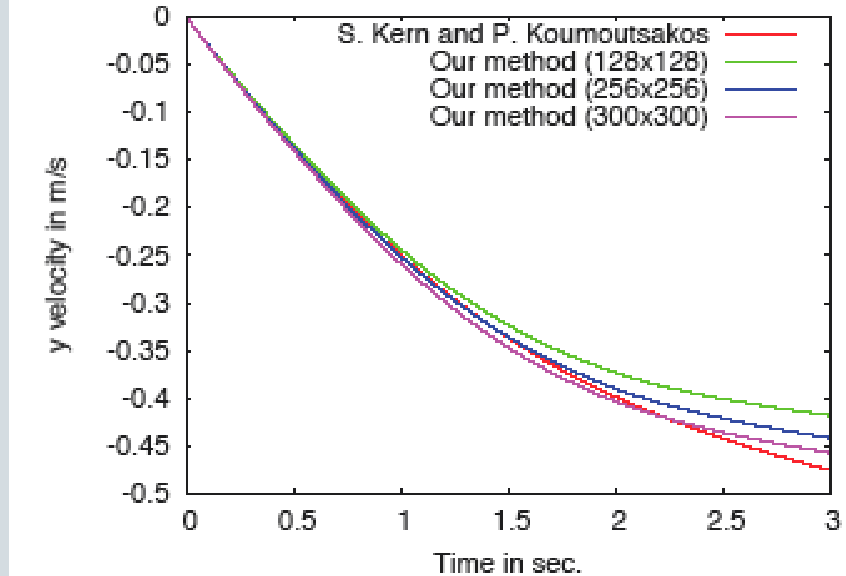

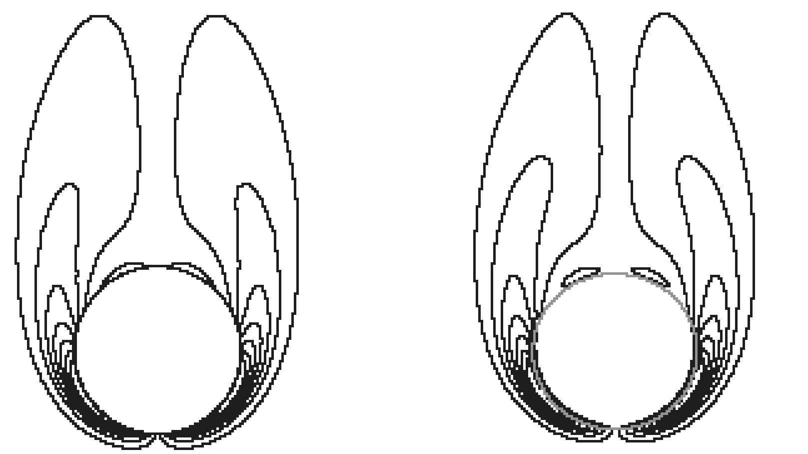

Validation against ALE methods and CPU times

Factor 10 speed-up over "conventional methods"

Sequence	Grid Resolution	Duration	Time Steps	CPU Time / Step
Cup 1	$100 \times 100 \times 100$	10 s	1300	8.60 s
Cup 2	$64 \times 64 \times 64$	10 s	1000	2.44 s
Spheres	$68 \times 24 \times 292$	4 s	400	5.96 s
Pyramid	$80 \times 80 \times 80$	3 s	600	12.4 s

Other example of two-way fluid structure interaction: flapping and self locomotion

Re=10

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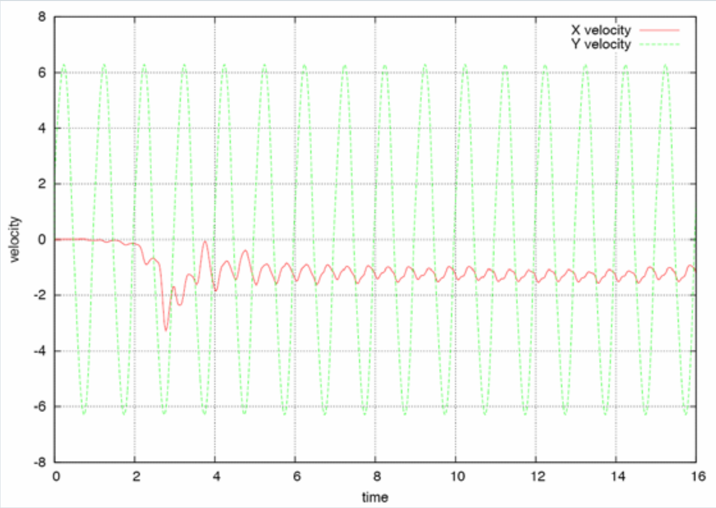

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