New Thin Piezoelectric Plate Models

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ABSTRACT: Early investigations on piezoelectric plates were based on a priori mechanical and experimental considerations. They assume plane stress and consider only transverse components of electric displacement and field. Beside, these were supposed constant in the plate thickness. Through an asymptotic analysis, this paper shows that mechanical hypotheses follow Kirchhoff-Love theory of thin plates. However, electric assumptions are found to be strongly dependent on the electric boundary conditions. That is, two regular problems should be distinguished: (1) the short circuited plate, for which only transverse electric displacement and field have to be considered—the electric potential is then found to be the sum of a known part, which depends on prescribed potentials, and an unknown part, which represents an induced potential and cannot be a priori neglected; the mechanical and electrical problems may be uncoupled; (2) the insulated plate, for which only in-plane electric displacement and field components are to be considered; the mechanical and electrical problems may be uncoupled for orthorhombic plates but not in general. Based on the above asymptotic analysis, two variational and local two-dimensional static models are presented for heterogeneous anisotropic plates. They are then applied to homogeneous and orthorhombic piezoelectric plates. For homogeneous orthorhombic piezoelectric plates, the electromechanical problem can be uncoupled. Hence, a mechanical problem is first solved for the mechanical displacement, then electric potentials are explicitly deduced from this displacement. Classical finite element codes having multilayer plate facilities can be used for solving the plate problems obtained.

INTRODUCTION

Piezoelectric materials find wide use in deformations and motion of structure detection, as well as in active structural control. They are either bonded or embedded in the structure in order to measure strains and displacements (sensing effect) or to provide localized strains through which the deformation of the structure can be controlled (actuation effect). Effective sensing and actuation of the resulting smart structure need that its electroelastic behavior be well modeled.

The concept of smart structure was first validated analytically and experimentally on beam elements, and rapidly structural problems of piezoelectric plates arose in modeling flat piezoelectric sensors and actuators. These were mainly analyzed with approximate theories (Destuynder et al., 1992; Tzou, 1993; Drozdov and Kalamkarov, 1996), based on assumed simplifying, a priori, hypotheses concerning directions of the electric field and displacement, and the representative form of the electric potential and even the constitutive behavior (Drozdov and Kalamkarov, 1996). Indeed, only transverse components of electric displacement and field are retained. The electric potential is generally considered linear through thickness, and sometimes constant. Besides, for most actuation applications, the induced potential due to the direct piezoelectric effect is often supposed negligible compared to the imposed potential (Destuynder et al., 1992; Tzou, 1993). Classical structural theories are also often adapted and the presence of piezoelectric elements is taken into account by a thermal analogy approach through the introduction of prescribed strains (Destuynder et al., 1992). But, in this way, the full coupled electroelastic behavior of smart structures is not completely modeled, since the mass and stiffness of the piezoelectrics are not considered. Therefore, good representations of the piezoelectric effect have been achieved using either additional electronic circuits that take into account the induced potential, as in self-sensing actuation techniques (Dosch, Inman and Garcia, 1992) and shunted piezoelectrics (Hagood and von Flotow, 1991), or using through-thickness quadratic variation of the electric potential as proposed by Rogacheva (1994) using an asymptotic analysis.

By the beginning of this decade, several asymptotic theories were proposed to deal with piezoelectric plates. They differ in the scaling techniques adopted and their results depend on the type of electric boundary conditions applied on upper and lower faces of the plate. Most common conditions, in technical applications, are either of Neumann (prescribed surface electric charges) or Dirichlet (imposed surface electric potentials) type. Maugin and Attou (1992) used asymp-

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totic integration theory, in its variational form, that exploits the "zoom technique," to establish the first two orders of an asymptotic theory of thin piezoelectric plates in the framework of electrostatics. At the first-order of approximation, a purely mechanical Love-Kirchhoff theory emerged, while the electric potential satisfies a two-dimensional Poisson-Neumann problem, with an effective dielectric constant accounting for electromechanical coupling. Nevertheless, these results do not agree with the thin-plate limit obtained by Bisegna and Maceri from exact three-dimensional solutions (Bisegna and Maceri, 1996a), and with the consistent piezoelectric plate theory they derived (Bisegna and Maceri, 1996b). In particular, the deflection given by Maugin and Attou (1992) depends on elastic constants and not on piezoelectric and dielectric ones, contradicting results of Bisegna and Maceri (1996a, 1996b). The procedure used in Bisegna and Maceri (1996b) to derive field equations governing the piezoelectric problem is based on the initial functions method in conjunction with a re-scaling of the applied loads. A piezoelectric plate model was also derived by Rogacheva (1994) as a zero-curvatures shell model. The latter was based on a priori assumptions from an asymptotic analysis similar to that used by Maugin and Attou (1992) for isolated piezoelectric plates. Similar results were then obtained. However, Rogacheva (1994) has also considered tangential polarization and short-circuited plates.

The major difficulty introduced by Dirichlet-type electric boundary conditions, besides electromechanical coupling, is the non-homogeneity (in the mathematical sense). To deal with this problem in piezoelectric shells, Bernadou and Haenel (1995a, 1995b) simply used a variable change in order to make the electric boundary conditions homogeneous. They take their inspiration from the classical handling of mechanical Dirichlet boundary conditions. Canon and Lenczner (1994) and Lenczner (1996) used Lagrange's multipliers to deal with electric Dirichlet conditions applied to multilayer elastic plates containing piezoelectric inclusions (Canon and Lenczner, 1994) or including distributed piezoelectric actuators and a distributed electric circuit (Lenczner, 1996). Their plate models were derived as the limit of the three-dimensional problem when the thickness vanishes in the framework of asymptotic methods. The electric potential and displacement were found to be independent of the transverse coordinate. Besides, electromechanical couplings exist between in-plane components of electric field and stresses on one hand, and between the jump of the electric potential on the upper and lower faces and plane stresses, on the other hand. However, these results do not agree with those of Bisegna and Maceri (1996b) who found that the transverse component of the electric displacement is linear in the thickness direction and that the electric potential has quadratic law variation with respect to thickness coordinate, even in the case of thin piezoelectric plates.

Recently, Saravanos, Heyliger and Hopkins (1997) proposed layer-wise (or discrete layer) theories using piecewise linear continuous approximations along the thickness direction for both displacement and electric potential fields. These theories can model both global and local electromechanical responses of smart composite laminates. Finite element formulations with added electric potential degrees of freedom were also developed for quasi-static and dynamic analyses of smart composite structures containing piezoelectric layers. Numerical analyses indicate that electric fields exist in the piezoelectric layers even with closed-circuit conditions (zero potential). Moreover, figures show that the induced electric potential is parabolic inside piezoelectric layers and vanishes at their upper and lower skins.

It is the objective of this paper to present an asymptotic theory for piezoelectric plates in the presence of Dirichlet electric boundary conditions. These are taken into account by considering the electric field induced by the imposed electric potentials. That is, the electric potential is decomposed into a known component, completely defined from prescribed electric potentials and an unknown component representing an induced electric potential. It will be shown that, for the bending problem, this induced potential is proportional to bending strains and its contribution in the system behavior appears through a modification of the constitutive elastic equations and an additional electric force. Associate numerical handling is then made easier, since standard finite elements could be used provided that modified elastic constants and electric force could be entered (Rahmoune et al., 1996; Rahmoune and Osmont, 1996). It is found here that the consistent hypothesis for transverse electric displacement and field for thin layers is "the electric displacement and field are respectively constant and linear through thickness," contrary to above asymptotic theories where it is generally assumed or derived that "the electric displacement and field are respectively constant through thickness." Moreover, it is found that the mechanical problem is of a Kirchhoff-Love type.

In the following, the paper focuses on developing Kirchhoff-Love models for piezoelectric plates using an asymptotic approach extending to the piezoelectric media the theory of Ciarlet and Destuynder (1979) devoted to elastic media. Numerical implementation was described in Rahmoune et al. (1996), Rahmoune and Osmont (1996), and detailed in Rahmoune (1997). Local three-dimensional equations and their associated variational formulation will first be recalled. Next, two-dimensional variational and local equations of short-circuited and insulated piezoelectric plates will be deduced as a limit when the aspect ratio of the plate vanishes. Applications are then made to homogeneous and orthotropic plates.

THREE DIMENSIONAL PROBLEM

In the three-dimensional space, of reference frame $(o, x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon)$, let us consider a homogeneous piezoelectric body Ω^ε, of thickness 2ε and characterized by its elastic, piezoelectric and dielectric constants, C^{ε}_{ijkl}, e^{ε}_{kij} and $\in^{\varepsilon}_{ij}$ respectively. $\partial\Omega^{\varepsilon}$ denotes its regular boundary which is decomposed into upper, lower and lateral surfaces, $\Sigma^{+\varepsilon}$, $\Sigma^{-\varepsilon}$ and Γ^{ε}, respectively. The latter is clamped on Γ^{ε}_0 and free on its complementary part Γ^{ε}_1. The middle surface of the piezoelectric body is denoted ω and has a boundary $\partial\omega = \gamma$, split into a clamped part γ_0 and a free complementary part γ_1. The domain can carry mechanical body forces f^{ε} and surface forces $g^{\pm\varepsilon}$ on upper and lower faces $\Sigma^{\pm\varepsilon}$. Electrically, it could be either short-circuited, through prescribed potentials $V^{\pm\varepsilon}$ on $\Sigma^{\pm\varepsilon}$ and surface charge q^{ε} on Γ^{ε}, or insulated, i.e., charge-free on $\partial\Omega^{\varepsilon}$.

Local Equations

Local three-dimensional problems consist of finding mechanical displacement u^{ε} and electric potential φ^{ε} satisfying the following equations:

- *mechanical equilibrium and boundary conditions:*

$$-\sigma^{\varepsilon}_{ij,j} = f^{\varepsilon}_i \quad \text{in } \Omega^{\varepsilon} \tag{1}$$

$$\sigma^{\varepsilon}_{ij} n_j = g^{\pm\varepsilon}_i \quad \text{on } \Sigma^{\pm\varepsilon}, \qquad \sigma^{\varepsilon}_{ij} n_j = 0 \quad \text{on } \Gamma^{\varepsilon}_1, \qquad u^{\varepsilon}_i = 0 \quad \text{on } \Gamma^{\varepsilon}_0 \tag{2}$$

where σ^{ε} is the symmetric Cauchy stress tensor, and n_j are outward normal components,

- *electric equilibrium and boundary conditions:*

$$D^{\varepsilon}_{i,i} = 0 \quad \text{in } \Omega^{\varepsilon} \tag{3}$$

$$\varphi^{\varepsilon} = V^{\pm\varepsilon} \quad \text{on } \Sigma^{\pm\varepsilon}, \qquad D^{\varepsilon}_{\alpha} n_{\alpha} = -q^{\varepsilon} \quad \text{on } \Gamma^{\varepsilon} \tag{4}$$

where D^{ε} is the electric displacement vector,

- *linear piezoelectric constitutive equations:*

$$\sigma^{\varepsilon}_{ij} = C^{\varepsilon}_{ijkl}\varepsilon_{kl}(u^{\varepsilon}) - e^{\varepsilon}_{k;ij}E_k(\varphi^{\varepsilon}), \qquad D^{\varepsilon}_i = e^{\varepsilon}_{i;kl}\varepsilon_{kl}(u^{\varepsilon}) + \in^{\varepsilon}_{ik} E_k(\varphi^{\varepsilon}) \tag{5}$$

representing the converse (actuator) and direct (sensor) piezoelectric effects. Notice that these are expressed using the short-circuited elastic, piezoelectric and dielectric constants. In Equation (5), ε and E are the symmetric linear strain tensor and electric field vector related to the mechanical displacement u^{ε} and electric potential φ^{ε}, respectively, through,

$$\varepsilon_{kl}(u^{\varepsilon}) = \frac{1}{2}(u^{\varepsilon}_{k,l} + u^{\varepsilon}_{l,k}), \qquad E_k(\varphi^{\varepsilon}) = -\varphi^{\varepsilon}_{,k} \tag{6}$$

It is worthwhile to notice that two important regular problems could be distinguished according to the electric boundary conditions considered. The first corresponds to the short-circuited piezoelectric media, and the second to the electrically insulated piezoelectric body, for which the potential is completely unknown, in particular on the boundary. The natural electric boundary condition is then, assumed to be, for the latter case,

$$D^{\varepsilon}_i n_i = 0 \quad \text{on } \partial\Omega^{\varepsilon} \tag{7}$$

Variational Formulation

The three-dimensional variational problem, associated to local equations (1–5), consists of finding the couple $(u^{\varepsilon}, \varphi^{\varepsilon})$ in the admissible space

$$V(\Omega^{\varepsilon}) = \{(v^{\varepsilon}, \psi^{\varepsilon}) \in [H^1(\Omega^{\varepsilon})]^3 \times H^1(\Omega^{\varepsilon}), \quad v^{\varepsilon} = 0 \ \text{on } \Gamma^{\varepsilon}_0, \quad \psi^{\varepsilon} = V^{\pm\varepsilon} \ \text{on } \Sigma^{\pm\varepsilon}\}$$

satisfying the variational equation:

$$\int_{\Omega^{\varepsilon}} \sigma^{\varepsilon}_{ij}(u^{\varepsilon}, \varphi^{\varepsilon})\varepsilon_{ij}(v^{\varepsilon})d\Omega^{\varepsilon} + \int_{\Omega^{\varepsilon}} D^{\varepsilon}_i(u^{\varepsilon}, \varphi^{\varepsilon})E_i(\psi^{\varepsilon})d\Omega^{\varepsilon}$$

$$= \int_{\Omega^{\varepsilon}} f^{\varepsilon}_i v^{\varepsilon}_i d\Omega^{\varepsilon} + \int_{\Sigma^{\pm\varepsilon}} g^{\pm\varepsilon}_i v^{\varepsilon}_i d\Sigma^{\pm\varepsilon} + \int_{\Gamma^{\varepsilon}} q^{\varepsilon}\psi^{\varepsilon} d\Gamma^{\varepsilon} \tag{8}$$

$$\forall (v^{\varepsilon}, \psi^{\varepsilon}) \in V_0(\Omega^{\varepsilon})$$

where $V_0(\Omega^{\varepsilon}) = \{(v^{\varepsilon}, \psi^{\varepsilon}) \in [H^1(\Omega^{\varepsilon})]^3 \times H^1(\Omega^{\varepsilon}), v^{\varepsilon} = 0 \text{ on } \Gamma^{\varepsilon}_0, \psi^{\varepsilon} = 0 \text{ on } \Sigma^{\pm\varepsilon}\}$. This variational problem has a unique solution (Rahmoune, 1997).

ASYMPTOTIC ANALYSIS

The asymptotic development technique (Ciarlet and Destuynder, 1979) is adapted here to get equivalent Love-Kirchhoff piezoelectric plate models as limits of the above three-dimensional variational problem when ε tends to vanish.

In order to apply the asymptotic analysis, the domain Ω^{ε} is first made fixed; i.e., independent of ε through the following geometrical variable change:

$$x^{\varepsilon} = (x^{\varepsilon}_1, x^{\varepsilon}_2, x^{\varepsilon}_3) \in \Omega^{\varepsilon} \rightarrow (x_1, x_2, \varepsilon x_3) \in \Omega$$

$$\Omega = \omega \times]-1,+1[\tag{9}$$

Then mechanical and electric parameters scaling are defined. The former is classically treated, as detailed in Ciarlet and Destuynder (1979). However, the latter depends on electric boundary conditions.

Mechanical Displacement and Forces Scaling

Geometrical considerations lead to the following a priori estimates: $u_i^\varepsilon \in H^1(\Omega^\varepsilon)$ is a priori associated to $u_i(\varepsilon) \in H^1(\Omega)$ such that

$$u_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 u_\alpha(\varepsilon)(x), \quad u_3^\varepsilon(x^\varepsilon) = \varepsilon u_3(\varepsilon)(x) \quad (10)$$

Similar relations hold for $v_i^\varepsilon \in H^1(\Omega^\varepsilon)$ associated, a priori, to $v_i(\varepsilon) \in H^1(\Omega)$. It is convenient for mathematical purposes to introduce the following scaling on body and surface forces defined as

$$\begin{aligned} f_\alpha^\varepsilon(x^\varepsilon) &= \varepsilon^2 f_\alpha(x), \quad f_3^\varepsilon(x^\varepsilon) = \varepsilon^3 f_3(x); \\ g_\alpha^\varepsilon(x^\varepsilon) &= \varepsilon^3 g_\alpha(x), \quad g_3^\varepsilon(x^\varepsilon) = \varepsilon^4 g_3(x) \end{aligned} \quad (11)$$

where f and g are independent of ε. Material constants are supposed independent of ε, i.e.,

$$\begin{aligned} C_{ijkl}^\varepsilon(x^\varepsilon) &= C_{ijkl}(x_1, x_2, x_3), \quad e_{k;ij}^\varepsilon(x^\varepsilon) = e_{k;ij}(x_1, x_2, x_3), \\ \in_{ij}^\varepsilon(x^\varepsilon) &= \in_{ij}(x_1, x_2, x_3) \end{aligned} \quad (12)$$

Electric Potential Scaling for the Insulated Plate

The plate is assumed insulated, i.e., Equation (7) holds. Thus, the variational problem (8) reduces to finding $(u^\varepsilon, \varphi^\varepsilon) \in V_{01}(\Omega^\varepsilon)$, so that,

$$\begin{aligned} &\int_{\Omega^\varepsilon} [\varepsilon_{ij}(v^\varepsilon) C_{ijkl}^\varepsilon \varepsilon_{kl}(u^\varepsilon) - \varepsilon_{ij}(v^\varepsilon) e_{k;ij}^\varepsilon E_k(\varphi^\varepsilon)] d\Omega^\varepsilon \\ &+ \int_{\Omega^\varepsilon} [E_i(\psi^\varepsilon) e_{i;kl}^\varepsilon \varepsilon_{kl}(u^\varepsilon) + E_i(\psi^\varepsilon) \in_{ik}^\varepsilon E_k(\varphi^\varepsilon)] d\Omega^\varepsilon \\ &= \int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon d\Omega^\varepsilon + \int_{\Sigma^{\pm\varepsilon}} g_i^{\pm\varepsilon} v_i^\varepsilon d\Sigma^{\pm\varepsilon}, \quad \forall (v^\varepsilon, \psi^\varepsilon) \in V_{01}(\Omega^\varepsilon) \end{aligned} \quad (13)$$

where $V_{01}(\Omega^\varepsilon) = \{(v^\varepsilon, \psi^\varepsilon) \in [H^1(\Omega^\varepsilon)]^3 \times H^1(\Omega^\varepsilon),\ v^\varepsilon = 0 \text{ on } \Gamma_0^\varepsilon\}$.

The order of ε in the unknown potential φ^ε is selected such that the variational problem (13) has an asymptotic solution. This is found to be in the form (Rahmoune, 1997):

$$\varphi^\varepsilon = \varepsilon^2 \varphi(\varepsilon) \quad (14)$$

Similar relations hold for ψ^ε associated to $\psi(\varepsilon)$.

Electric Potential and Charge Scaling for the Short-Circuited Plate

The plate is now assumed short-circuited, i.e., electric boundary conditions (4) hold. Prior to the direct application of the asymptotic development technique, non-homogeneous Dirichlet electric boundary conditions (4.1) should be made homogeneous. To this end, the electric potential φ^ε is split into a linear part $\varphi^{0\varepsilon}$, known through the given potentials on $\Sigma^{\pm\varepsilon}$, and an unknown part ϕ^ε, so that,

$$\varphi^\varepsilon = \varphi^{0\varepsilon} + \phi^\varepsilon, \quad (\varphi^{0\varepsilon}, \phi^\varepsilon) \in V_K(\Omega^\varepsilon) \times V_U(\Omega^\varepsilon) \quad (15)$$

where

$$V_K(\Omega^\varepsilon) = \left\{ \frac{U^{+\varepsilon} + U^{-\varepsilon}}{2} + \frac{U^{+\varepsilon} - U^{-\varepsilon}}{2\varepsilon} x_3, \forall U^{\pm\varepsilon} \in H^{1/2}(\Sigma^{\pm\varepsilon}) \right\}$$

$$V_U(\Omega^\varepsilon) = \{\psi^\varepsilon \in H^1(\Omega^\varepsilon), \quad \psi^\varepsilon = 0 \text{ on } \Sigma^{\pm\varepsilon}\}$$

Hence, the variational problem (8), reduces to finding $(u^\varepsilon, \phi^\varepsilon) \in V_0(\Omega^\varepsilon)$, such that:

$$\begin{aligned} &\int_{\Omega^\varepsilon} [\varepsilon_{ij}(v^\varepsilon) C_{ijkl}^\varepsilon \varepsilon_{kl}(u^\varepsilon) + E_i(\psi^\varepsilon) e_{i;kl}^\varepsilon \varepsilon_{kl}(u^\varepsilon)] d\Omega^\varepsilon \\ &+ \int_{\Omega^\varepsilon} [-\varepsilon_{ij}(v^\varepsilon) e_{k;ij}^\varepsilon E_k(\phi^\varepsilon) + E_i(\psi^\varepsilon) \in_{ik}^\varepsilon E_k(\phi^\varepsilon)] d\Omega^\varepsilon \\ &= \int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon d\Omega^\varepsilon + \int_{\Sigma^{\pm\varepsilon}} g_i^{\pm\varepsilon} v_i^\varepsilon d\Sigma^{\pm\varepsilon} + \int_{\Gamma^\varepsilon} q^\varepsilon \psi^\varepsilon d\Gamma^\varepsilon \\ &- \int_{\Omega^\varepsilon} [-\varepsilon_{ij}(v^\varepsilon) e_{k;ij}^\varepsilon E_k(\varphi^{0\varepsilon}) + E_i(\psi^\varepsilon) \in_{ik}^\varepsilon E_k(\varphi^{0\varepsilon})] d\Omega^\varepsilon \\ &\forall (v^\varepsilon, \psi^\varepsilon) \in V_0(\Omega^\varepsilon) \end{aligned} \quad (16)$$

The second integral in the left hand side (l.h.s.) of Equation (16) is the contribution of the unknown potential ϕ^ε, shown here to represent the induced potential, often neglected in the literature. However, the last integral in the r.h.s. of Equation (16) is the equivalent electric force due to the imposed potential $\varphi^{0\varepsilon}$. The above variational formulation can be used either for actuator or sensor analysis. But, in the latter case, the last two integrals in Equation (16) should be dropped, according to Equation (7). This is more detailed in the next sub-section.

Now, scaling orders of $\varphi^{0\varepsilon}$, ϕ^ε and q^ε are chosen so that an asymptotic solution exists for the variational problem (16). These are found to be in the form (Rahmoune, 1997),

$$\phi^\varepsilon = \varepsilon^2 \phi(\varepsilon), \quad \varphi^{0\varepsilon} = \varepsilon^3 \varphi^0(\varepsilon); \quad q^\varepsilon = \varepsilon^2 q \quad (17)$$

where q is independent of ε. Using Equation (15), the total potential is then taken in the form,

$$\varphi^\varepsilon = \varepsilon^3 \varphi(\varepsilon), \quad \varphi(\varepsilon) = \varphi^0(\varepsilon) + \varepsilon^{-1} \phi(\varepsilon) \quad (18)$$

Similar relations hold for ψ^ε associated, a priori, to $\psi(\varepsilon)$ which can be also decomposed into a known and an unknown part as in Equations (15), (17) and (18). Notice that Equations (14), (17) and (18) ensure that the corresponding electric parameters appear in the zeroth order of the asymptotic solution.

SHORT-CIRCUITED PIEZOELECTRIC PLATE MODEL

This section aims to present the limit variational problem associated with a short-circuited piezoelectric plate when ε tends to zero. Starting from the three-dimensional variational problem (16), in conjunction with mechanical [Equations (10)–(12)] and electric [Equations (15,17,18)] parameters scaling, the asymptotic mechanical displacement field and electric potential are first deduced through an asymptotic analysis; then a limit two-dimensional variational problem is defined, together with its associated local equations.

Asymptotic Mechanical Displacement and Electric Potential

According to the mechanical displacement scaling Equation (10), the strains can be scaled as

$$\varepsilon^{\varepsilon}_{\alpha\beta} = \varepsilon^2\varepsilon_{\alpha\beta}(\varepsilon), \quad \varepsilon^{\varepsilon}_{\alpha 3} = \varepsilon\varepsilon_{\alpha 3}(\varepsilon), \quad \varepsilon^{\varepsilon}_{33} = \varepsilon_{33}(\varepsilon) \tag{19}$$

It can be shown (Rahmoune, 1997) that, when ε goes to zero, we have

$$\varepsilon_{\alpha\beta} \rightarrow \varepsilon_{\alpha\beta}(0), \quad \varepsilon_{i3} \rightarrow 0 \tag{20}$$

which naturally leads to the following asymptotic displacement,

$$\begin{aligned} u_\alpha(x_1, x_2, x_3) &= \xi_\alpha(x_1, x_2) - x_3\partial_\alpha\xi_3(x_1, x_2) \\ u_3(x_1, x_2, x_3) &= \xi_3(x_1, x_2) \end{aligned} \tag{21}$$

where $(\xi_\alpha,\xi_3) \in V_{KL} \times W_{KL}$ with $V_{KL} = \{\xi_\alpha \in H^1(\omega), \xi_\alpha = 0 \text{ on } \gamma_0\}$, $W_{KL} = \{\xi_3 \in H^1(\omega), \xi_3 = \partial_\nu\xi_3 = 0 \text{ on } \gamma_0\}$.

From the electric potential scaling (18), the electric field scaling has the form

$$\begin{aligned} E_\alpha(\varphi^\varepsilon) = \varepsilon^3 E_\alpha(\varphi(\varepsilon)), \quad E_3(\varphi^\varepsilon) = \varepsilon^2 E_3(\varphi(\varepsilon)) \\ E_i(\varphi(\varepsilon)) = E_i(\varphi^0(\varepsilon)) + \varepsilon^{-1}E_i(\phi(\varepsilon)) \end{aligned} \tag{22}$$

It can be shown (Rahmoune, 1997) that when ε tends to zero

$$\varphi^0(\varepsilon) \rightarrow \varphi^0(0), \quad \frac{1}{3}\phi(\varepsilon) \rightarrow \phi^* \tag{23}$$

Consequently the electric potential tends to φ such that:

$$\varphi = \varphi^0 + \phi^* \tag{24}$$

where

$$(\varphi^0, \phi^*) \in V_L \times V_Q$$

and

$$V_L = \{\varphi^0 \in H^1(\Omega), \quad \varphi^0 = V^\pm \text{ on } \Sigma^\pm\}$$

$$V_Q = \left\{\psi \in H^1(\Omega),\ \psi = 0 \text{ on } \Sigma^\pm,\ \psi = -\int_{-1}^{x_3} E_3(\psi)\,dx_3 \text{ and } \int_{-1}^{+1} E_3(\psi)\,dx_3 = 0\right\}$$

It is worthwhile noticing that Equations (20) and (23) are weak convergences. It was shown that these are also strong convergences (see Rahmoune, 1997 for details).

These results suggest the following comments:

- The asymptotic mechanical displacement field (21) is of the Kirchhoff-Love type ($\varepsilon_{i3} = 0$).
- The asymptotic electric potential (24) has a known part φ^0 entirely defined by the known potentials on the upper and lower faces, and an unknown part ϕ^* corresponding to the induced potential. ϕ^* is often neglected in the literature (Rahmoune et al., 1996).
- Only transverse components of the electric field are of importance, which is often retained in the literature, but finds here a mathematical justification [Equation (23)].

Asymptotic Electric Displacement and Stresses

In order to satisfy the electric equilibrium (3) and boundary conditions (4.2), in-plane and transverse electric displacement components are scaled separately:

$$D^{\varepsilon}_3 = \varepsilon^2 D_3(\varepsilon), \quad D^{\varepsilon}_\alpha = \varepsilon D_\alpha(\varepsilon) \tag{25}$$

where

$$D_3(\varepsilon) = e_{3;kl}\kappa_{kl}(\varepsilon) + \in_{3k} F_k(\phi(\varepsilon))$$

$$D_\alpha(\varepsilon) = \varepsilon[e_{\alpha;kl}\kappa_{kl}(\varepsilon) + \in_{\alpha k} F_k(\phi(\varepsilon))] = \varepsilon\overline{D}_\alpha(\varepsilon)$$

and

$$\kappa_{\alpha\beta}(\varepsilon) = \varepsilon_{\alpha\beta}(\varepsilon), \quad \kappa_{\alpha 3}(\varepsilon) = \varepsilon^{-1}\varepsilon_{\alpha 3}(\varepsilon)$$

$$\kappa_{33}(\varepsilon) = \varepsilon^{-2}\varepsilon_{33}(\varepsilon), \quad F_\alpha(\varepsilon) = \varepsilon E_\alpha(\varphi(\varepsilon))$$

$$F_3(\varepsilon) = E_3(\varphi(\varepsilon))$$

$\varphi(\varepsilon)$ is defined in Equation (24). Since $\kappa_{kl}(\varepsilon)$ and $F_k(\phi(\varepsilon))$ are bounded, then when ε tends to zero, electric displacement components converge weakly to

$$D_3(\varepsilon) \rightarrow D_3(0), \quad D_\alpha(\varepsilon) \rightarrow 0 \tag{26}$$

Classical scaling of the stresses is defined as (Ciarlet and Destuynder, 1979),

$$\sigma^{\varepsilon}_{\alpha\beta} = \varepsilon^2\sigma_{\alpha\beta}(\varepsilon), \quad \sigma^{\varepsilon}_{\alpha 3} = \varepsilon^3\sigma_{\alpha 3}(\varepsilon), \quad \sigma^{\varepsilon}_{33} = \varepsilon^4\sigma_{33}(\varepsilon) \tag{27}$$

where

$$\sigma_{\alpha\beta}(\varepsilon) = C_{\alpha\beta kl}\kappa_{kl}(\varepsilon) - e_{k;\alpha\beta}F_k(\varepsilon)$$

$$\sigma_{\alpha 3}(\varepsilon) = \varepsilon^{-1}[C_{\alpha 3kl}\kappa_{kl} - e_{k;\alpha 3}F_k] = \varepsilon^{-1}\bar{\sigma}_{\alpha 3}(\varepsilon)$$

$$\sigma_{33}(\varepsilon) = \varepsilon^{-2}[C_{33kl}\kappa_{kl} - e_{k;33}F_k] = \varepsilon^{-2}\bar{\sigma}_{33}(\varepsilon)$$

Using Equations (19), (22), (25) and (27), the three-dimensional variational problem (8) becomes:

$$\int_{\Omega}[\sigma_{\alpha\beta}(\varepsilon)\varepsilon_{\alpha\beta}(v(\varepsilon)) + \bar{\sigma}_{\alpha 3}(\varepsilon)\varepsilon^{-1}\varepsilon_{\alpha 3}(v(\varepsilon))$$

$$+ \bar{\sigma}_{33}(\varepsilon)\varepsilon^{-2}\varepsilon_{33}(v(\varepsilon))]d\Omega$$

$$+\int_{\Omega}[\bar{D}_{\alpha}(\varepsilon)\varepsilon E_{\alpha}(\psi(\varepsilon)) + D_3(\varepsilon)E_3(\psi(\varepsilon))]d\Omega$$

$$= \int_{\Omega} f_i v_i d\Omega + \int_{\Sigma^{\pm}} g_i^{\pm} v_i d\Sigma^{\pm} + \int_{\Gamma} \varepsilon q\psi(\varepsilon)d\Gamma \tag{28}$$

where $\psi(\varepsilon) = \psi^0(\varepsilon) + \varepsilon^{-1}\chi(\varepsilon)$.

Keeping in mind that $\kappa_{ij}(\varepsilon)$ are bounded, then

1. Multiplying Equation (28) by ε, setting $v = v_{\alpha}$ and $\psi = 0$, and tending ε to zero
2. Multiplying Equation (28) by ε^2, setting $v = v_3$ and $\psi = 0$, and tending ε to zero, weak [shown to be also strong (Rahmoune, 1997)] convergence of $\sigma_{i3}(\varepsilon)$ to $\sigma_{i3}(0)$ is obtained with:

$$\bar{\sigma}_{\alpha 3}(0) = \bar{\sigma}_{33}(0) = 0 \tag{29}$$

These results indicate that:

- Only the transverse component of the electric displacement is of importance [Equation (26)], in-plane components may be neglected as it is the case for the electric field; this is a classical hypothesis, but finds its justification here.
- Transverse stresses σ_{i3} may also be neglected compared to in-plane components [Equation (29)].

Two-Dimensional Variational Problem

Let's set $v_{\alpha}(x_1, x_2, x_3) = \eta_{\alpha}(x_1, x_2) - x_3\partial_{\alpha}\eta_3(x_1, x_2)$, $v_3(x_1, x_2, x_3) = \eta_3(x_1, x_2)$ and $\psi = 0$ in the variational problem (28) and ε tends to zero, in order to obtain

$$\int_{\Omega}\sigma_{\alpha\beta}(0)\varepsilon_{\alpha\beta}(v)\,d\Omega = \int_{\Omega} f_i v_i d\Omega + \int_{\Sigma^{\pm}} g_i^{\pm} v_i d\Sigma^{\pm} \tag{30}$$

where

$$\varepsilon_{\alpha\beta}(v) = \gamma_{\alpha\beta} - x_3\partial_{\alpha\beta}\eta_3$$

and $\sigma_{\alpha\beta}(0)$ is the limit of $\sigma_{\alpha\beta}(\varepsilon)$, given by,

$$\sigma_{\alpha\beta}(0) = \bar{C}_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}(u) - \bar{e}_{3;\alpha\beta}[E_3(\varphi^0) + E_3(\phi^*)] \tag{31}$$

in which

$$\bar{C}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} - C_{3j\gamma\delta}S_{3j3k}C_{k3\alpha\beta}$$

$$\bar{e}_{3;\alpha\beta} = e_{3;\alpha\beta} - C_{3j\alpha\beta}S_{3j3k}e_{3;k3}$$

are modified elastic and piezoelectric constants due to $\sigma_{i3} = 0$. The 3×3 matrix (S_{3j3k}) is the inverse of the 3×3 matrix (C_{3j3k}).

Stress resultants $N_{\alpha\beta}$ and moments $M_{\alpha\beta}$ are now introduced:

$$N_{\alpha\beta} = \int_{-1}^{+1}\sigma_{\alpha\beta}(0)dx_3, \quad M_{\alpha\beta} = \int_{-1}^{+1}x_3\sigma_{\alpha\beta}(0)dx_3 \tag{32}$$

then included in Equation (30) to get the following two-dimensional variational problem:

$$\int_{\omega}[N_{\alpha\beta}(\xi, \varphi)\gamma_{\alpha\beta}(\eta) - M_{\alpha\beta}(\xi, \varphi)\partial_{\alpha\beta}\eta_3]d\omega$$

$$= \int_{\omega}[p_i\eta_i - m_{\alpha}\partial_{\alpha}\eta_3]d\omega, \quad \forall(\eta_{\alpha}, \eta_3) \in V_{KL} \times W_{KL} \tag{33}$$

where

$$p_i = \int_{-1}^{+1} f_i dx_3 + (g_i^+ + g_i^-)$$

$$m_{\alpha} = \int_{-1}^{+1} x_3 f_{\alpha} dx_3 + (g_{\alpha}^+ - g_{\alpha}^-)$$

To prove the importance of the unknown potential ϕ^*, we shall first show that the unknown electric field $E_3(\phi^*)$ [Equation (31)] may be expressed in terms of bending strains. To this end, let us set $v = 0$ and $\psi^0 = 0$ in Equation (28), then multiplying by ε, and make ε vanish to obtain

$$\int_{\Omega}[e_{3;kl}\kappa_{kl}(\xi) + \epsilon_{33}F_3(\phi^*)]F_3(\chi)d\Omega = 0 \quad \forall\chi \in V_Q \tag{34}$$

From Equations (27) and (29), κ_{i3} can be expressed in terms of $\kappa_{\alpha\beta}$ and F_3,

$$\kappa_{\gamma 3} = -S_{\gamma 3k3}[C_{k3\gamma\delta}\kappa_{\gamma\delta} - e_{3;k3}F_3]$$
$$\kappa_{33} = -S_{33k3}[C_{k3\gamma\delta}\kappa_{\gamma\delta} - e_{3;k3}F_3] \tag{35}$$

These are included in Equation (34), which is integrated by parts in order to get

$$\partial_3[\bar{e}_{3;\alpha\beta}\kappa_{\alpha\beta} + \overline{\epsilon}_{33}F_3(\phi^*)] = 0 \quad \text{in } \Omega \tag{36}$$

where

$$\overline{\epsilon}_{33} = \epsilon_{33} - e_{3;j3}S_{3j3k}e_{3;k3}$$

$\overline{\epsilon}$ values are modified dielectric constants due to $\sigma_{i3} = 0$. From the above equation, the unknown electric field $E_3(\phi^*)$ can be written in terms of bending strains as:

$$E_3(\phi^*) = \frac{1}{\overline{\epsilon}_{33}}\int_{-1}^{x_3}\bar{e}_{3;\alpha\beta}\partial_{\alpha\beta}\xi_3 dy_3 + cst \tag{37}$$

The constant of integration is determined through the condition $\int_{-1}^{+1}E_3(\phi^*)dx_3 = 0$, leading to the final expression of the unknown electric field,

$$E_3(\phi^*) = \frac{1}{\overline{\epsilon}_{33}}\int_{-1}^{x_3}\bar{e}_{3;\alpha\beta}\partial_{\alpha\beta}\xi_3 dy_3 - \frac{1}{2}\int_{-1}^{+1}\left[\frac{1}{\overline{\epsilon}_{33}}\int_{-1}^{x_3}\bar{e}_{3;\alpha\beta}\partial_{\alpha\beta}\xi_3 dy_3\right]dx_3 \tag{38}$$

Hence, the unknown electric potential can be simply defined as

$$\phi^*(x_3) = -\int_{-1}^{x_3}E_3(\phi^*)dy_3 \tag{39}$$

since $\phi^* = 0$ on $\Sigma^{\pm}$. Unknown electric field (38) and potential (39) are interpreted as induced electric field and potential. These are often neglected in the literature (Rahmoune et al., 1996).

Expression (38) of the induced electric field is now substituted in the constitutive equation (31), which becomes,

$$\sigma_{\alpha\beta}(0) = \sigma^*_{\alpha\beta}(0) - \bar{e}_{3;\alpha\beta}E_3(\varphi^0) \tag{40}$$

where

$$\sigma^*_{\alpha\beta}(0) = \overline{C}_{\alpha\beta\lambda\delta}\gamma_{\delta\lambda} - C^*_{\alpha\beta\lambda\delta}(x_3)\partial_{\delta\lambda}\xi_3$$

and

$$C^*_{\alpha\beta\delta\lambda} = x_3\overline{C}_{\alpha\beta\delta\lambda} + \bar{e}_{3;\alpha\beta}\left[\frac{1}{\overline{\epsilon}_{33}}\int_{-1}^{x_3}\bar{e}_{3;\delta\lambda}dy_3 - \frac{1}{2}\int_{-1}^{+1}\left(\frac{1}{\overline{\epsilon}_{33}}\int_{-1}^{x_3}\bar{e}_{3;\delta\lambda}dy_3\right)dx_3\right]$$

Using definitions (32) of the normal and moment stress resultants, we get,

$$N_{\alpha\beta}(\xi,\varphi^0) = N^*_{\alpha\beta}(\xi) - N^0_{\alpha\beta}(\varphi^0)$$
$$M_{\alpha\beta}(\xi,\varphi^0) = M^*_{\alpha\beta}(\xi) - M^0_{\alpha\beta}(\varphi^0) \tag{41}$$

where $N^0_{\alpha\beta}$ and $M^0_{\alpha\beta}$ are related to the applied potential φ^0:

$$N^0_{\alpha\beta}(\varphi^0) = \int_{-1}^{+1}\bar{e}_{3;\alpha\beta}E_3(\varphi^0)dx_3$$
$$M^0_{\alpha\beta}(\varphi^0) = \int_{-1}^{+1}x_3\bar{e}_{3;\alpha\beta}E_3(\varphi^0)dx_3 \tag{42}$$

$N^*_{\alpha\beta}$ and $M^*_{\alpha\beta}$ include the effect of induced electric field. They are linked to membrane and bending strains through the following modified constitutive equations:

$$N^*_{\alpha\beta} = C^m_{\alpha\beta\delta\lambda}\gamma_{\delta\lambda} - C^{mb*}_{\alpha\beta\delta\lambda}\partial_{\delta\lambda}\xi_3$$
$$M^*_{\alpha\beta} = C^{bm}_{\alpha\beta\delta\lambda}\gamma_{\delta\lambda} - C^{b*}_{\alpha\beta\delta\lambda}\partial_{\delta\lambda}\xi_3 \tag{43}$$

where $C^m_{\alpha\beta\delta\lambda}$, $C^{mb*}_{\alpha\beta\delta\lambda}$, $C^{bm}_{\alpha\beta\delta\lambda}$, $C^{b*}_{\alpha\beta\delta\lambda}$ are membrane, membrane-bending and bending modified elastic plane stress constants. They are given by:

$$C^m_{\alpha\beta\delta\lambda} = \int_{-1}^{+1}\overline{C}_{\alpha\beta\delta\lambda}dx_3, \quad C^{mb*}_{\alpha\beta\delta\lambda} = \int_{-1}^{+1}C^*_{\alpha\beta\delta\lambda}dx_3$$
$$C^{bm}_{\alpha\beta\delta\lambda} = \int_{-1}^{+1}x_3\overline{C}_{\alpha\beta\delta\lambda}dx_3, \quad C^{b*}_{\alpha\beta\delta\lambda} = \int_{-1}^{+1}x_3C^*_{\alpha\beta\delta\lambda}dx_3 \tag{44}$$

It's worthwhile noticing that modified constitutive equations (43) are not symmetric due to membrane-bending elastic constant because of the piezoelectric coupling.

Substituting Equations (43) into (41), then in the variational problem (33), gives the following new variational property:

$$\int_\omega[N^*_{\alpha\beta}(\xi)\gamma_{\alpha\beta}(\eta) - M^*_{\alpha\beta}(\xi)\partial_{\alpha\beta}\eta_3]d\omega = \int_\omega[p_i\eta_i - m_\alpha\partial_\alpha\eta_3]d\omega + \int_\omega[N^0_{\alpha\beta}(\varphi^0)\gamma_{\alpha\beta}(\eta) - M^0_{\alpha\beta}(\varphi^0)\partial_{\alpha\beta}\eta_3]d\omega \tag{45}$$

Compared to a classical elastic variational problem, the above equation, indicates that the piezoelectric effect induces a modification of the constitutive equations according to Equation (43) and an extra generalized electric force/moment vector represented by the last r.h.s. integral in Equation (45). Hence, the numerical implementation of Equation (45) would be very easy provided that Equations (42) and (44) could be entered into a classical structural software. The solution of the above variational problem gives the mechanical displacement ξ.

Now let's see how to derive electric quantities such as ϕ^* or $E_3(\phi^*)$. For this, let $v = 0$ in the variational problem (28), which becomes,

$$\int_\Omega [D_3(\varepsilon)E_3(\psi) + D_\alpha(\varepsilon)E_\alpha(\psi)]d\Omega = \int_\Gamma \varepsilon q(\varepsilon)\psi d\Gamma \quad \forall\psi \tag{46}$$

Taking account of Equation (26) and making ε vanish, leads to:

$$\int_\Omega D_3(0)E_3(\psi)\,d\Omega = 0, \quad \forall\psi \in V_Q \tag{47}$$

which implies that $D_{3,3}(0) = 0$ in Ω. Hence, using Equations (25) and (35), we get:

$$D_3(0) = \bar{e}_{3;\alpha\beta}\gamma_{\alpha\beta} + \overline{\in}_{33}E_3(\varphi^0) \tag{48}$$

In summary, to get the solution of the electromechanical coupled problem for a smart short-circuited plate, the variational problem (45) is first solved in order to obtain the mechanical displacement, then the electric displacement, field and potential could be computed, a posteriori, by Equations (48), (38) and (39) respectively. Beside, the piezoelectric effect induces a modification of the constitutive equations, as indicated in Equation (43) and an extra electrical force term as defined in Equation (42). Here it was found that the induced electric potential and field are proportional to bending strains, whereas the electric displacement is constant through the thickness.

Local Problem

Integrating by parts the variational equation (33), gives the following local equations:

- *membrane problem*

$$\begin{aligned} -\partial_\beta N_{\alpha\beta} &= p_\alpha \quad \text{in } \omega \\ N_{\alpha\beta}\nu_\beta &= 0 \quad \text{on } \gamma_1 \end{aligned} \tag{49}$$

where $N_{\alpha\beta}$ are given by Equations (41)–(44) and ν_β are normal components of the outward normal unit vector to γ_1.

- *bending problem*

$$\begin{aligned} -\partial_{\alpha\beta}M_{\alpha\beta} &= p_3 + \partial_\alpha m_\alpha \quad \text{in } \omega \\ (\partial_\alpha M_{\alpha\beta})\upsilon_\beta + \partial_\tau(M_{\alpha\beta}\upsilon_\alpha\tau_\beta) &= -(\partial_\alpha m_\alpha)\nu_\alpha \quad \text{on } \gamma_1 \\ M_{\alpha\beta}\upsilon_\beta\upsilon_\alpha &= 0 \quad \text{on } \gamma_1 \end{aligned} \tag{50}$$

- *electric problem*

$$\partial_3 D_3 = 0 \quad \text{in } \omega \tag{51}$$

Discussion of the Short-Circuited Piezoelectric Plate Model

The present two-dimensional formulation shows that in-plane electric displacement and field do not appear. Hence, only their transverse components and in-plane stresses are retained in this model. These are usual hypotheses in the literature, but find here a mathematical background.

It was found that, when the induced potential is not neglected, electric displacement and field are constant and linear in the thickness direction, respectively. When the induced field is substituted in the constitutive equations, by its explicit expression in terms of the bending strains, the electromechanical problem uncouples. Thus, the mechanical problem is solved first for mechanical displacements, then these are used to compute, a posteriori, the electric quantities (field, potential, charge, displacement).

If the induced electric potential is neglected, for consistency reasons, the transverse electric field component is constant in the thickness direction. Consequently, the transverse electric displacement becomes linear, i.e., $D_3(0) = \bar{e}_{3;\delta\lambda}(\gamma_{\delta\lambda} - x_3\partial_{\delta\lambda}\xi_3) + \overline{\in}_{33}E_3(\varphi^0)$, but does not ensure the electric equilibrium equation (51). Hence, assuming constant transverse electric field and displacement, as retained in many papers, is contradictory. The right electric assumptions for short-circuited piezoelectric plate should be constant electric displacement and linear electric field as explained above.

According to Equation (48), bending strains do not induce electric charge, only membrane strains can do so for homogeneous piezoelectrics. However, both strains can generate electric charges for the practical case of multilayer configurations where piezoelectrics are either surface-mounted to or embedded in a host structure.

INSULATED PIEZOELECTRIC PLATE MODEL

The objective of this section is to develop the limit variational problem associated with an insulated piezoelectric plate when ε tends to zero. Starting from local equations (1)–(3) and (5)–(7) and their corresponding three-dimensional variational problem (13), in conjunction with mechanical [Equations (10)–(12)] and electric [Equation (14)] parameters scaling, asymptotic electric potential and displacement are first deduced through an asymptotic analysis; then an equivalent two-dimensional variational model is formulated together with its associated local equations.

Asymptotic Electric Displacement and Potential

Compared to the above short-circuited model, scaling of electric parameters are no longer those obtained previously, since the electric potential is now completely unknown. Its scaling is given by Equation (14), i.e.,

$$\varphi^{\varepsilon} = \varepsilon^2 \varphi(\varepsilon) \quad (52)$$

Hence, in-plane and transverse components of the electric field can be scaled as,

$$E_\alpha(\varphi^\varepsilon) = \varepsilon^2 E_\alpha(\varphi(\varepsilon)), \quad E_3(\varphi^\varepsilon) = \varepsilon E_3(\varphi(\varepsilon)) \quad (53)$$

Scaling of in-plane and transverse components of the electric displacement are now taken so as to satisfy the electric equilibrium equation (3) and boundary condition (7) :

$$D_\alpha^\varepsilon = \varepsilon^2 D_\alpha(\varepsilon), \quad D_3^\varepsilon = \varepsilon^3 D_3(\varepsilon) \quad (54)$$

where

$$D_\alpha(\varepsilon) = \bar{e}_{\alpha;kl}\kappa_{kl}(\varepsilon) + \bar{\in}_{\alpha k} F_k(\varepsilon)$$

$$D_3(\varepsilon) = \varepsilon^{-1}[\bar{e}_{3;kl}\kappa_{kl}(\varepsilon) + \bar{\in}_{3k} F_k(\varepsilon)] = \varepsilon^{-1}\bar{D}_3(\varepsilon)$$

and

$$F_\alpha(\varepsilon) = E_\alpha(\varphi(\varepsilon)), \quad F_3(\varepsilon) = \varepsilon^{-1}E_3(\varphi(\varepsilon))$$

Using Equations (19), (27), (53) and (54), the three-dimensional variational problem (13), written on the fixed domain, is now,

$$\int_\Omega [\sigma_{\alpha\beta}(\varepsilon)\varepsilon_{\alpha\beta}(v(\varepsilon)) + \bar{\sigma}_{\alpha 3}(\varepsilon)\varepsilon^{-1}\varepsilon_{\alpha 3}(v(\varepsilon))$$
$$+ \bar{\sigma}_{33}(\varepsilon)\varepsilon^{-2}\varepsilon_{33}(v(\varepsilon))]d\Omega + \int_\Omega [D_\alpha(\varepsilon)E_\alpha(\psi(\varepsilon))$$
$$+ \bar{D}_3(\varepsilon)\varepsilon^{-1}E_3(\psi(\varepsilon))]d\Omega = \int_\Omega f_i v_i d\Omega + \int_{\Sigma^\pm} g_i^\pm v_i d\Sigma^\pm \quad (55)$$

Contrary to the short-circuited plate model, it may be shown that the asymptotic electric potential is now constant in the plate thickness (Rahmoune, 1997), i.e., when ε goes to zero,

$$\varphi(\varepsilon) \to \varphi^0(x_1, x_2) \quad (56)$$

Besides, in-plane components of the electric displacement converge weakly according to:

$$D_\alpha(\varepsilon) \to D_\alpha(0) \quad \text{in } L^2 \quad (57)$$

when ε is made to vanish (Rahmoune, 1997). Here, in-plane electric displacement components are dominant; D_3E_3, the transverse electric work is negligible versus $D_\alpha E_\alpha$.

Two-Dimensional Variational Problem

Let's set $v_\alpha(\varepsilon) = \eta_\alpha(x_1, x_2) - x_3\partial_\alpha\eta_3(x_1, x_2)$, $v_3(\varepsilon) = \eta_3(x_1, x_2)$ and $\psi(\varepsilon) = \psi(x_1, x_2)$ in the variational problem (55) and tend ε to zero, in order to get,

$$\int_\Omega [\sigma_{\alpha\beta}(0)\varepsilon_{\alpha\beta}(v) + D_\alpha(0)E_\alpha(\psi)]d\Omega$$
$$= \int_\Omega f_i v_i d\Omega + \int_{\Sigma^\pm} g_i^\pm v_i d\Sigma^\pm \quad (58)$$

where

$$\varepsilon_{\alpha\beta}(v) = \gamma_{\alpha\beta} - x_3\partial_{\alpha\beta}\eta_3$$

$$\sigma_{\alpha\beta}(0) = \underline{C}_{\alpha\beta\delta\lambda}\varepsilon_{\delta\lambda} - \underline{e}_{\lambda;\alpha\beta}E_\lambda(\varphi)$$

$\underline{C}_{\alpha\beta\delta\lambda}$, $\underline{e}_{\lambda;\alpha\beta}$ are modified elastic and piezoelectric constants due to $\sigma_{i3} = 0$ and $\bar{D}_{3,3} = 0$. They are given in the Appendix.

Introducing the following electric charge:

$$Q_\lambda = \int_{-1}^{+1} D_\lambda dx_3 \quad (59)$$

where

$$D_\lambda = \underline{e}_{\lambda;\alpha\beta}\varepsilon_{\alpha\beta} + \underline{\in}_{\lambda\alpha}E_\lambda$$

($\underline{\in}_{\lambda\alpha}$ are modified dielectric constants given in the Appendix), together with stress resultants (32) and generalized forces (33) in the above three-dimensional variational problem, reduces it to the following two-dimensional one,

$$\int_\omega [N_{\alpha\beta}\gamma_{\alpha\beta}(\eta) - M_{\alpha\beta}\partial_{\alpha\beta}\eta_3 + Q_\lambda E_\lambda(\psi)]d\omega$$
$$= \int_\omega [p_i\eta_i - m_\alpha\partial_\alpha\eta_3]d\omega$$
$$\forall(\eta_\alpha, \eta_3, \psi) \in V_{KL} \times W_{KL} \times H^1(\omega)/R \quad (60)$$

This equation indicates that the electromechanical problem cannot be uncoupled in general, and that the coupling is between in-plane components of stresses and the electric field. However for orthorhombic piezoelectric media like PZT, the electrical and mechanical problems are not coupled.

Local Problem

Integrating by parts the above two-dimensional variational problem leads to the following local equations:

- *membrane problem*

$$\begin{aligned} -\partial_\beta N_{\alpha\beta} &= p_\alpha \quad \text{in } \omega \\ N_{\alpha\beta}\nu_\beta &= 0 \quad \text{on } \gamma_1 \end{aligned} \quad (61)$$

- *bending problem*

$$-\partial_{\alpha\beta}M_{\alpha\beta} = p_3 + \partial_\alpha m_\alpha \quad \text{in } \omega$$

$$(\partial_\alpha M_{\alpha\beta})\upsilon_\beta + \partial_\tau(M_{\alpha\beta}\upsilon_\alpha\tau_\beta) = -(\partial_\alpha m_\alpha)\nu_\alpha \quad \text{on } \gamma_1$$

$$M_{\alpha\beta}\upsilon_\beta\upsilon_\alpha = 0 \quad \text{on } \gamma_1 \tag{62}$$

- *electric problem*

$$\partial_\alpha Q_\alpha = 0 \quad \text{in } \omega$$

$$Q_\alpha\upsilon_\alpha = 0 \quad \text{on } \gamma_1 \tag{63}$$

Discussion of the Insulated Piezoelectric Model

The present theory shows that, in general, in-plane components of the electric field couples with all in-plane strains, leading to a coupling between mechanical and electrical phenomena.

The asymptotic electric potential was found to be independent of x_3 at the zeroth order in ε. This potential can be identified to the "mean value" of the electric potential through the thickness of the plate.

Moreover for orthorhombic piezoelectric plates, the electric and mechanical problems uncouple due to the constitutive laws. The electrical problem gives the "mean value" of the potential through the thickness of the plate. The mechanical problem, using insulated elastic constants, gives the displacements and as a consequence an additional electrical potential induced by the deformation.

APPLICATIONS

Mechanical membrane-bending coupling and electromechanical couplings depend on the symmetry and homogeneity of the piezoelectric plate. This section discusses these coupling phenomena for a homogeneous and orthorhombic piezoelectric media. In particular, such plates have no membrane-bending coupling.

Homogeneous Piezoelectric Plates

Transversely homogeneous piezoelectric medium has uniform through thickness elastic, piezoelectric and dielectric constants, i.e., independent of coordinate x_3. Hence, previous results can be simplified. In particular the variational problems are symmetric.

SHORT-CIRCUITED PIEZOELECTRIC PLATE

The induced electric field defined by Equation (38), reduces to,

$$E_3(\phi^*) = x_3 \frac{\bar{e}_{3;\alpha\beta}}{\bar{\epsilon}_{33}} \partial_{\alpha\beta}\xi_3 \tag{64}$$

Thus, the induced potential (39) is now simply,

$$\phi^*(x_3) = \frac{1}{2}(1 - x_3^2)\frac{\bar{e}_{3;\alpha\beta}}{\bar{\epsilon}_{33}} \partial_{\alpha\beta}\xi_3 \tag{65}$$

Equations (64) and (65) show that both induced electric field and potential are proportional to bending strains. The former is linear in x_3, whereas the latter is parabolic.

From these relations, generalized constitutive equations (43) reduce to,

$$N^*_{\alpha\beta} = C^m_{\alpha\beta\lambda\delta}\gamma_{\lambda\delta}, \quad M^*_{\alpha\beta} = -C^{b*}_{\alpha\beta\lambda\delta}\partial_{\lambda\delta}\xi_3 \tag{66}$$

where

$$C^m_{\alpha\beta\lambda\delta} = 2\bar{C}_{\alpha\beta\lambda\delta}, \quad C^{b*}_{\alpha\beta\lambda\delta} = \frac{2}{3}C'_{\alpha\beta\lambda\delta}$$

$$C'_{\alpha\beta\lambda\delta} = \bar{C}_{\alpha\beta\lambda\delta} + \frac{\bar{e}_{3;\alpha\beta}\bar{e}_{3;\lambda\delta}}{\bar{\epsilon}_{33}}$$

according to Equation (44). Notice that only bending constants are modified by the piezoelectric effect. In fact $C'_{\alpha\beta\chi\delta}$ constants are insulated plane stress elastic constants.

Since there is no membrane-bending coupling, two uncoupled mechanical problems can be defined:

- a membrane problem, for which:

$$\int_\omega N^*_{\alpha\beta}(\xi_\alpha)\gamma_{\alpha\beta}(\eta_\alpha)\,d\omega$$

$$= \int_\omega p_\alpha\eta_\alpha d\omega + \int_\omega N^0_{\alpha\beta}(\varphi^0)\gamma_{\alpha\beta}(\eta_\alpha)\,d\omega, \quad \forall\eta_\alpha \in V_{KL} \tag{67}$$

- a bending problem, for which:

$$\int_\omega M^*_{\alpha\beta}(\xi_3)\,\partial_{\alpha\beta}\eta_3 d\omega = \int_\omega(-p_3\eta_3 + m_\alpha\partial_\alpha\eta_3)\,d\omega$$

$$+ \int_\omega M^0_{\alpha\beta}(\varphi^0)\partial_{\alpha\beta}\eta_3 d\omega, \quad \forall\eta_3 \in W_{KL} \tag{68}$$

The electric displacement is still defined by equation (48).

For constant prescribed electric field $E_3(\varphi^0)$, Equations (42) reduce to:

$$N^0_{\alpha\beta}(\varphi^0) = 2\bar{e}_{3;\alpha\beta}E_3(\varphi^0), \quad M^0_{\alpha\beta}(\varphi^0) = 0 \tag{69}$$

Therefore, the piezoelectric coupling induces modifications of the classical plate model:

- a membrane constitutive law using short-circuited plane stress elastic constants

- a bending constitutive law using insulated plane stress elastic constants
- additional generalized electric in-plane forces $N^0_{\alpha\beta}(\varphi^0)$ proportional to the prescribed electric field $E_3(\varphi^0)$ for the membrane problem
- no additional generalized electric moments $M^0_{\alpha\beta}(\varphi^0)$ relative to the prescribed electric field $E_3(\varphi^0)$ but an additional potential proportional to bending

So, to solve the problem for the short-circuited thin plate, one calculates the prescribed electric potential φ^0, then the displacements ξ of the plate with modified constitutive laws, then the induced potential ϕ^*. If it is desired to solve the variational problem using finite element codes, one can have trouble introducing different elastic constants for membrane and bending. This can be overcome by replacing the piezoelectric plate by an equivalent three layered plate having usual constants for plates.

INSULATED PIEZOELECTRIC PLATE

The two-dimensional electromechanical variational problem is that defined in Equation (60), for which generalized constitutive equations read now:

$$N_{\alpha\beta} = 2[\underline{C}_{\alpha\beta\lambda\delta}\gamma_{\lambda\delta} - \underline{e}_{\lambda;\alpha\beta}E_\lambda(\varphi)]$$
$$M_{\alpha\beta} = -\frac{2}{3}\underline{C}_{\alpha\beta\lambda\delta}\partial_{\lambda\delta}\xi_3 \qquad (70)$$

$$Q_\alpha = 2[\underline{e}_{\alpha;\lambda\delta}\gamma_{\lambda\delta} + \underline{\in}_{\alpha\lambda}E_\lambda(\varphi)] \qquad (71)$$

Here, the electromechanical variational problem (60) does not uncouple. Moreover the coupling is between the in-plane components of membrane strain or stresses and plane components of electric field or displacement. From Equations (69), (70) and (71) it is clear that bending strains could not be correlated to any electric quantity measured on the plate surface.

Orthorhombic Homogeneous Piezoelectric Plates

Actually, commonly used piezoelectric materials are transverse isotropic (like PZT) or orthorhombic (like PVDF). That is the reason why we focus here on orthorhombic piezoelectric materials for which simplifications occur. These materials have the following short-circuited elastic matrix,

$$C = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ C_{1122} & C_{2222} & C_{2233} & 0 & 0 & 0 \\ C_{1133} & C_{2233} & C_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1212} \end{bmatrix}$$

and piezoelectric and dielectric matrices,

$$e = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{1;31} & 0 \\ 0 & 0 & 0 & e_{2;32} & 0 & 0 \\ e_{3;11} & e_{3;22} & e_{3;33} & 0 & 0 & 0 \end{bmatrix}$$

$$\in = \begin{bmatrix} \in_{11} & 0 & 0 \\ 0 & \in_{22} & 0 \\ 0 & 0 & \in_{33} \end{bmatrix}$$

Piezoelectric material constants depend on the direction of the material axes and electric polarization. Here, the former is supposed to be parallel to the reference axes and the latter is supposed perpendicular to the plate. Modified material constants, taking account of the piezoelectric effect in a classic mechanical problem, are now presented.

SHORT-CIRCUITED PIEZOELECTRIC PLATE

1. For an extension problem, the plane stress elastic constants are,

$$\bar{C}_{\alpha\beta\lambda\delta} = C_{\alpha\beta\lambda\delta} - \frac{C_{33\lambda\delta}C_{33\alpha\beta}}{C_{3333}} \qquad (72)$$

   where C_{ijkl} are the short-circuited elastic constants.
2. For a bending problem, the plane stress elastic constants are

$$C'_{\alpha\beta\lambda\delta} = \bar{C}_{\alpha\beta\lambda\delta} + \frac{\bar{e}_{3;\lambda\delta}\bar{e}_{3;\alpha\beta}}{\bar{\in}_{33}} = C^i_{\alpha\beta\lambda\delta} - \frac{C^i_{33\lambda\delta}C^i_{33\alpha\beta}}{C^i_{3333}} \qquad (73)$$

   where

$$\bar{e}_{3;\alpha\beta} = e_{3;\alpha\beta} - \frac{C_{33\alpha\beta}e_{3;33}}{C_{3333}}, \quad \bar{\in}_{33} = \in_{33} - \frac{e_{3;33}e_{3;33}}{\in_{33}}$$

   The constants $C^i_{\alpha\beta\lambda\delta}$ are the insulated elastic constants.

   The only difference with the homogeneous case is a simplification of the plane stress elastic constants.

INSULATED PIEZOELECTRIC PLATE

Since modified piezoelectric constants $\underline{e}_{\lambda;\alpha\beta}$ are nil here, there is no coupling between in-plane components of stresses and electric field. The mechanical problem and the electric problem are not coupled. One obtains a classical elastic plate model with plane stress elastic constants having for membrane and bending the expressions:

$$N_{\alpha\beta} = 2\underline{C}_{\alpha\beta\lambda\delta}\gamma_{\lambda\delta}, \quad M_{\alpha\beta} = -\frac{2}{3}\underline{C}_{\alpha\beta\lambda\delta}\partial_{\lambda\delta}\xi_3 \qquad (74)$$

where

$$\underline{C}_{\alpha\beta\lambda\delta} = C^i_{\alpha\beta\lambda\delta} - \frac{C^i_{33\lambda\delta}C^i_{33\alpha\beta}}{C^i_{3333}}$$

The electrical potential φ is then the solution of the following problem:

$$\Delta\varphi = 0 \quad \text{in } \omega, \qquad \frac{\partial\varphi}{\partial n} = 0 \quad \text{on } \gamma = \partial\omega \tag{75}$$

which can be used to find the constant potential. This result is similar to that given in Maugin and Attou (1992).

CONCLUSION

The present theory shows that, for thin homogeneous piezoelectric plates, a priori consistent mechanical hypotheses are:

1. Only in-plane stresses have to be retained ($\sigma_{3i} = 0$).
2. The transverse electric displacement is constant through the plate thickness.
3. Mechanical displacement field is of Kirchhoff-Love type; i.e., in-plane and transverse components are, respectively, linear and constant through thickness.
4. The electric potential is the sum of two potentials φ_e and φ_i:
   - The former φ_e, which vary linearly through the thickness of the plate, depends on the electrical boundary conditions.
   - The latter φ_i, which vary quadratically through the thickness of the plate, does not depend on the electrical boundary conditions.

However, a priori, other electric assumptions depend on the electric boundary conditions. They are:

- *for a short-circuited plate:*
  (a) Only transverse components of the electric field and displacement have to be retained.
  (b) The electric potential φ_e is known explicitly from the prescribed potentials on the upper and lower faces.
  (c) The electric potential φ_i is known explicitly from the flexural displacement and is called the induced potential.
- *for an insulated plate*
  (a) Only in-plane components of the electric field and displacement have to be retained.
  (b) The electric potential φ_e is constant through the thickness and is only coupled to the membrane displacement; moreover, for an orthorhombic plate, this potential does not depend on the displacement.
  (c) The electric potential φ_i is known explicitly from the flexural displacement and is called the induced potential.

The present theory also indicates that bending strains could not be correlated to any measurable or applied electric quantity on piezoelectric plate surfaces.

These results make it possible to develop a similar plate theory for laminates with some of the layers being piezoelectric. Moreover, classical finite element codes can be used to find approximations of multilayer plate problems with piezoelectric layers. In practice, it is possible to solve a mechanical problem for the displacements; then the potential is obtained explicitly from these displacements.

For plates continuously heterogeneous through thickness, the results are more complicated.

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APPENDIX

Modified Constants for the Insulated Piezoelectric Plate

Modified material constant used in Equations (58) and (59) are defined by,

$$\underline{C}_{\alpha\beta\delta\lambda} = C_{\alpha\beta\delta\lambda} - C_{\alpha\beta j3}\tilde{S}_{k3j3}\tilde{C}_{k3\delta\lambda} - e_{3;\alpha\beta}\tilde{h}_{3;\delta\lambda}$$

$$\underline{e}_{\lambda;\alpha\beta} = e_{\lambda;\alpha\beta} - C_{\alpha\beta k3}\tilde{S}_{k3j3}\tilde{e}_{\lambda;j3} - e_{3;\alpha\beta}\tilde{\in}_{3\lambda}$$

$$\underline{\in}_{\lambda\alpha} = \in_{\lambda\alpha} - e_{\alpha;k3}\tilde{S}_{k3j3}\tilde{e}_{\lambda;j3} - \in_{3\alpha}\tilde{\in}_{3\lambda}$$

where

$$\tilde{C}_{\alpha\beta k3} = C_{\alpha\beta k3} + \frac{e_{3;k3}e_{3;\alpha\beta}}{\in_{33}}$$

$$\tilde{S}_{k3j3} = \left[C_{k3j3} + \frac{e_{3;k3}e_{3;j3}}{\in_{33}}\right]^{-1}$$

$$\tilde{e}_{\lambda;j3} = e_{\lambda;j3} - \frac{e_{3;j3}\in_{3\lambda}}{\in_{33}}$$

$$\tilde{\in}_{3\lambda} = \frac{1}{\in_{33}}[\in_{3\lambda} + e_{3;k3}\tilde{S}_{k3j3}\tilde{e}_{\lambda;j3}]$$

$$\tilde{h}_{3;\delta\lambda} = -\frac{1}{\in_{33}}[e_{3;\delta\lambda} - e_{3;k3}\tilde{S}_{k3j3}\tilde{C}_{j3\delta\lambda}]$$

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