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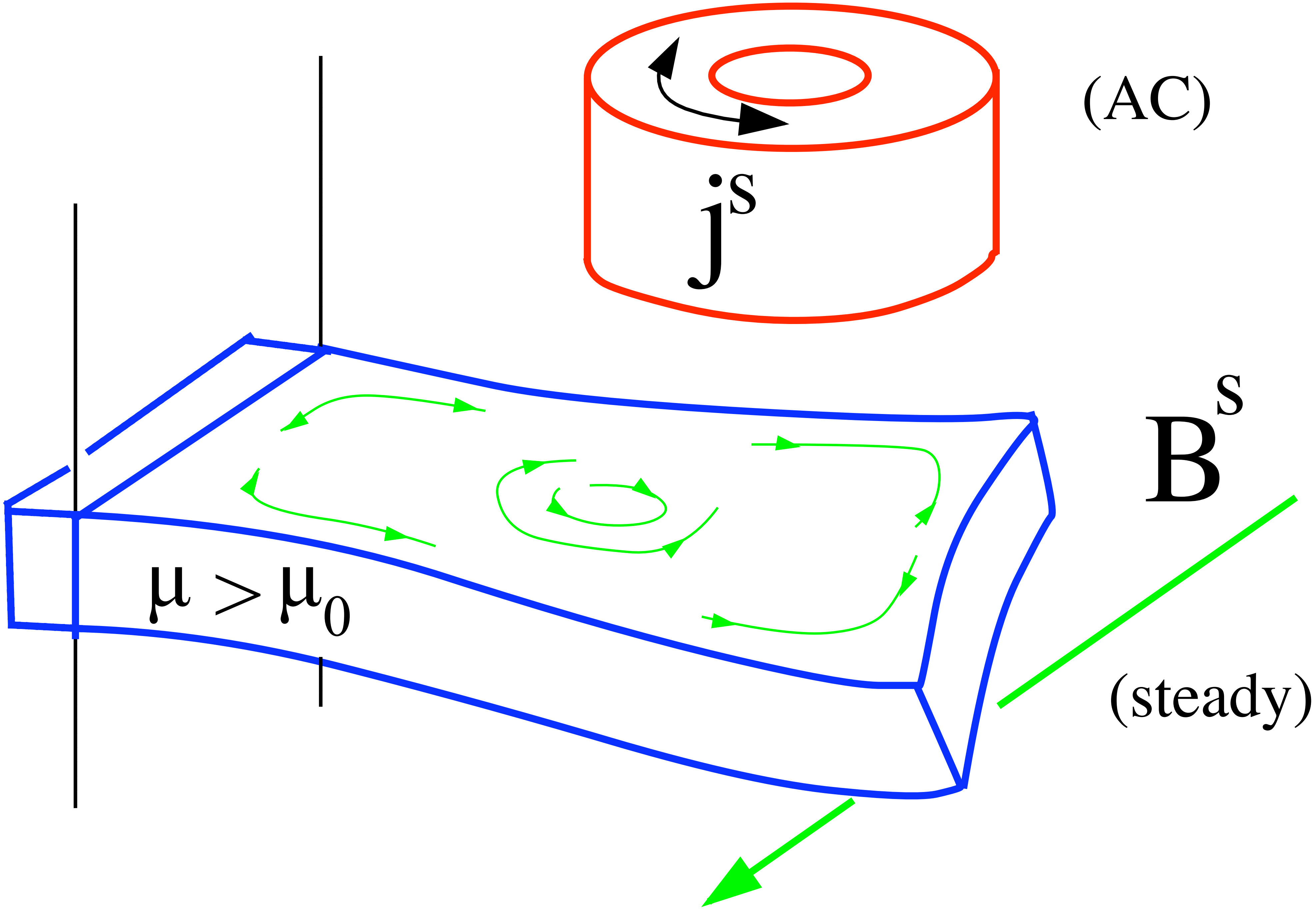
The long-standing question of forces in electromagnetics:

Forces inside magnetized matter

A standard benchmark (pbs. 12 &
16 of the "TEAM workshop")



(electric field contribution neglected)

$$\boxed{F = -\nabla p_0 - \frac{1}{8\pi} H^2 \nabla\mu + \frac{\mu}{c} J \times H}$$

L. Landau, E. Lifschitz: **Électrodynamique des milieux continus,** Mir (Moscou), 1965, p. 190

(for a fluid; p_0 is pressure)

$$\boxed{F = M \cdot \nabla B + M \times (\nabla \times B) - \nabla B \cdot (v \times (M \times v))/c^2}$$

P. Penfield, Jr., H.A. Haus: **Electrodynamics of Moving Media,**
The MIT Press (Cambridge, Ma.), 1967, p. 216

$$F = (J \times B) - \frac{1}{2} H^2 \nabla\mu$$

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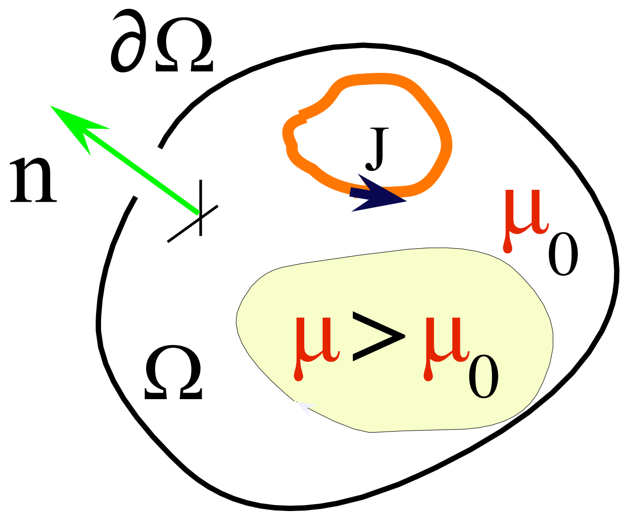
$$\boxed{\mathbf{M}\cdot\nabla\mathbf{H} = M^j \cdot \partial_j H^i} \leftarrow \text{Define} \rightarrow \boxed{\nabla\mathbf{H}\cdot\mathbf{M} \triangleq M^j \cdot \partial_i H^j}$$

(orthonormal reference frame, Einstein's convention)

$$\mathbf{M}\cdot\nabla\mathbf{H} - \nabla\mathbf{H}\cdot\mathbf{M} = \mathbf{M}\times \mathrm{rot}\,\mathbf{H}$$

$$\boxed{\mathbf{B} = \mu\mathbf{H} \equiv \mu_0\mathbf{H} + \mathbf{M}}$$



"Helmholtz force": $\mathbf{J}\times\mathbf{B} - \frac{1}{2}\mathbf{H}^2\nabla\mu$

"Kelvin force": $\mathbf{J}\times\mathbf{B} + \mathbf{M}\cdot\nabla\mathbf{H}$

"Maxwell force": $\mathbf{J}\times\mathbf{B} + \nabla\mathbf{H}\cdot\mathbf{M}$

same if rot $\mathbf{H} = 0$

"Magnetic charge force": $\mathbf{J}\times\mathbf{B} - \mathbf{H}\,\mathrm{div}\,\mathbf{M}$

$\int_\Omega \text{Maxw.} = \int_\Omega \text{Helm.};\quad \int_\Omega \text{MCF.} = \int_\Omega \text{Kelv.} = \int_\Omega \text{Helm.}$ if rot $\mathbf{H} = 0$

Another relation (among many)

$$\mathbf{B} = \mu\mathbf{H} \equiv \mu_0\mathbf{H} + \mathbf{M}$$

$$\boxed{\nabla\mathbf{H}\cdot\mathbf{M} + \frac{1}{2}\mathbf{H}^2\nabla\mu = \nabla((\mathbf{H}\cdot\mathbf{M})/2)}$$

i.e., 'Maxwell' – 'Helmholtz' $= \nabla((\mathbf{H}\cdot\mathbf{M})/2)$ whatever rot $\mathbf{H}$

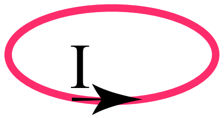
$$\int v \cdot \text{'Maxwell'} - \int v \cdot \text{''Helmholtz'} = -\int ((\mathbf{H}\cdot\mathbf{M})/2)\ \mathrm{div}\ v \quad \text{for all } v$$

same virtual work if $\mathrm{div}\ v = 0$

i.e., no difference for incompressible media

Origin of Maxwell and Kelvin forms

"Current loop"
magnetic dipole



$\mathbf{m} = \mu_0 I \text{ area}$

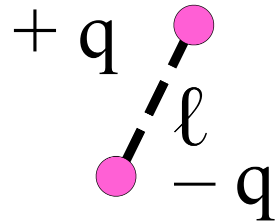
Force = $\nabla \mathbf{H} \cdot \mathbf{m}$

$(\partial_i H^j m^j)$

Maxwell

"Charge pair"
magnetic dipole

$\mathbf{m} = \mu_0 q/\ell$



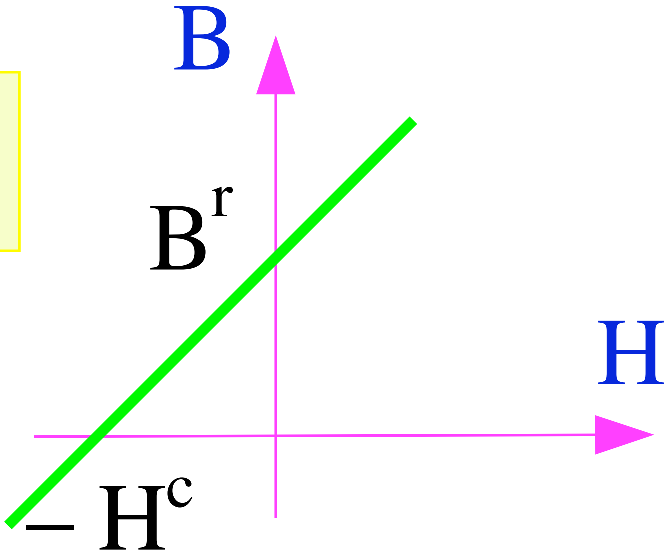
Force = $\mathbf{m} \cdot \nabla \mathbf{H}$

$(\partial_j H^i m^j)$

Kelvin

Magnetized matter as cloud of dipoles

$$\boxed{B = B^r + \mu_0 H}$$



$$\boxed{B = \mu_0 (H + H^c)}$$

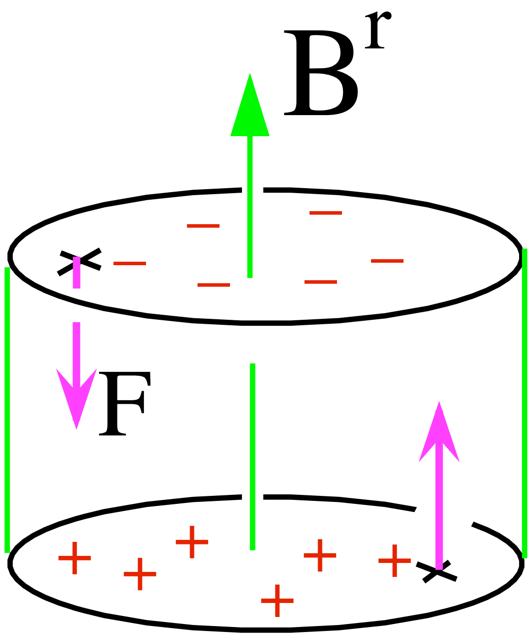
$(M = B^r \text{ here})$

$(M = \mu_0 H^c \text{ here})$

$$\mathrm{div}\, B = 0$$

$$B = B^r + \mu_0 H$$

$$\mathrm{rot}\, H = 0$$



$$\mathrm{div}\, \tilde{B} = \boxed{-\mathrm{div}\, B^r}$$

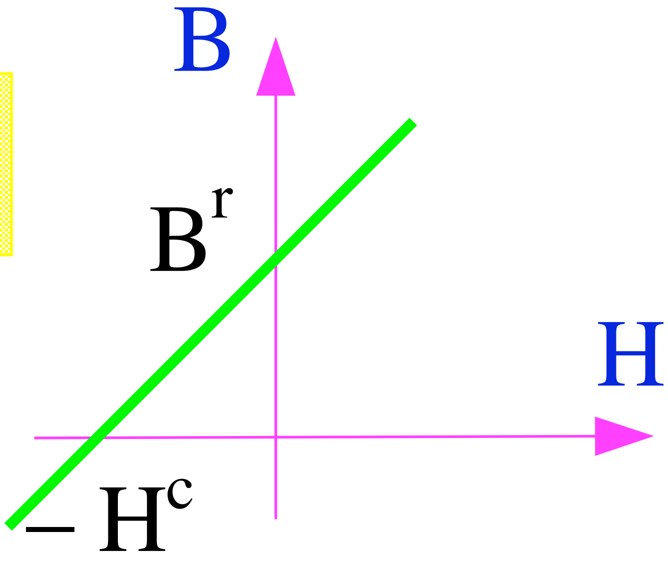
$$\tilde{B} = \mu_0 H$$

$$\mathrm{rot}\, H = 0$$

$$\boxed{F = -(\mathrm{div}\, B^r)\, H}$$

$$\boxed{B = B^{r} + \mu_0 H}$$

$(M = B^{r}$ here$)$



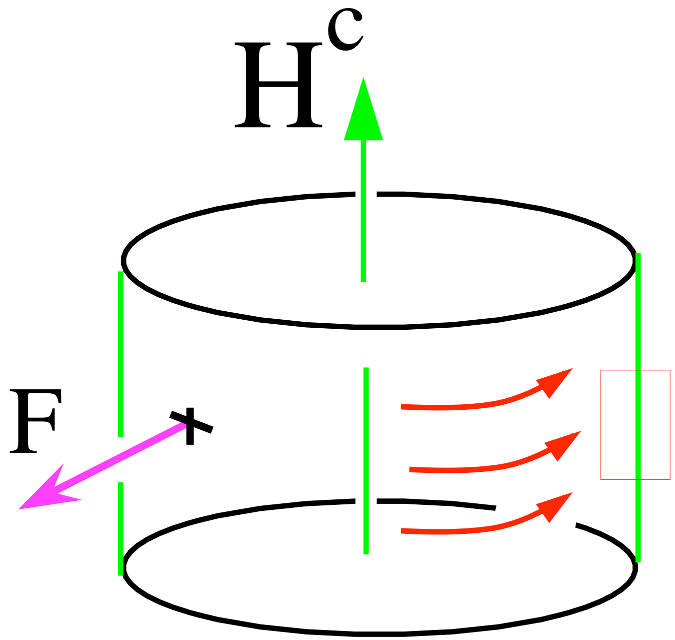
$$\boxed{B = \mu_0 (H + H^{c})}$$

$(M = \mu_0 H^{c}$ here$)$

$$\mathrm{div}\, B = 0$$

$$B = \mu_0 \tilde{H}$$

$$\mathrm{rot}\, \tilde{H} = \boxed{\mathrm{rot}\, H^{c}}$$



$$\mathrm{div}\, B = 0$$

$$B = \mu_0 (H + H^{c})$$

$$\mathrm{rot}\, H = 0$$

$$\boxed{F = (\mathrm{rot}\, H^{c}) \times B}$$

Proving Helmholtz right

Electromagnetics with motion

Neglect displacement currents and Coulomb forces

Kinematics:

$$x \longrightarrow x + u(t, x)$$

$$u(0, x) = 0$$

$$v = \partial_t u$$

$(u$ is "flow" of $v)$

Eddy-current equation:

$$\boxed{\begin{gathered} \operatorname{rot} H = \sigma_u (E + v \times B) + J^s \\ \partial_t B + \operatorname{rot} E = 0, \;\; H = \nu_u B \end{gathered}}$$

$$A(t) = A_0 - \int_0^t E(s)\, ds, \;\; B = \operatorname{rot} A$$

$$\sigma_u (\partial_t A - v \times \operatorname{rot} A) + \operatorname{rot}(\nu_u \operatorname{rot} A) = J^s, \; A(0) = A_0$$

$$\sigma_u(\partial_t A - v \times \mathrm{rot}\, A) + \mathrm{rot}(\nu_u\, \mathrm{rot}\, A) = J^s,\ A(0) = A_0$$

abridge as: $\sigma_u(\partial_t A - v \times B) + \partial_A \Psi(u,\ A) = J^s \quad (*)$

where $\Psi(u, A) = \frac{1}{2} \int \nu_u |\mathrm{rot}\, A|^2$

observe that

$$d_t[\Psi(u(t),\ A(t))] = \int \partial_u \Psi \cdot \underbrace{\partial_t u}_{= v} + \int \partial_A \Psi \cdot \partial_t A$$

and that

$$\underbrace{\sigma_u(\partial_t A - v \times B)}_{\equiv -J} \cdot \partial_t A = \sigma_u |\partial_t A - v \times B|^2 + v \cdot J \times B$$

Now, dot multiply (*) by $\partial_t A$ and get increase of stored energy

supplied power

$$\int \sigma_u |\partial_t A - v \times B|^2 + d_t \Psi(u, A) + \ldots$$

Joule

$$\ldots + \boxed{\int v \cdot (J \times B - \partial_u \Psi(u, A))} = -\int J^s \cdot E$$

Virtual power sent to elastic compartment of the system by its magnetic compartment

Hence (VPP), force must be

$$\boxed{f = J \times B - \partial_u \Psi(u, A)}$$

Force must be $f = J \times B - \partial_u \Psi(u, A)$

Confirmation (K is kinetic energy):

We found

$$\text{Joule} + d_t \Psi_{mag} + \int (J \times B - \partial_u \Psi_{mag}) \cdot \partial_t u = \int J^s \cdot \partial_t A$$

and energy conservation mandates

$$\boxed{\text{Joule} + d_t(\Psi_{mag} + \Psi_{ela} + K) = \int F \cdot \partial_t u + \int J^s \cdot \partial_t A}$$

where F is externally applied force. Subtract, get

$$d_t(\Psi_{ela} + K) = \int (F + J \times B - \partial_u \Psi_{mag}) \cdot \partial_t u$$

as if "magnetic force" f was added to "applied force" F

Computing $-\partial_u \Psi$

$$(\nu = 1/\mu;\ \ \nu_0 = 1/\mu_0)$$

$$\Psi(u, A) = \frac{1}{2}\int \nu_0 |\mathrm{rot}\ A|^2$$

No dependence on u. No other force than $J \times B$.

(Don't confuse v and ν. Color...)

Computing $-\partial_u \Psi$. If ν depends on x:

Set $u(t, x) = tv(x)$. Differentiate in t at $t =$ 0 with A fixed (because *partial* derivative):

$$\boxed{\Psi(tv, A) = \frac{1}{2} \int \nu_t \, |\mathrm{rot}\, A|^2}$$

It seems natural to set $\nu_t(x + tv(x)) = \nu(x)$

Then $\partial_t \nu_t + v \cdot \nabla \nu_t = 0$, hence $\partial_t \nu_t|_{t=0} = -v \cdot \nabla \nu$

Result: $f = |B|^2/2 \, \nabla \nu$ Helmholtz!

Discussing the assumption

Is $\nu_t(x + t\mathbf{v}(x)) = \nu(x)$ so natural?

- Pro: Same particle, same properties
- Con: Some properties depend on strain. Why not ν? E.g.,

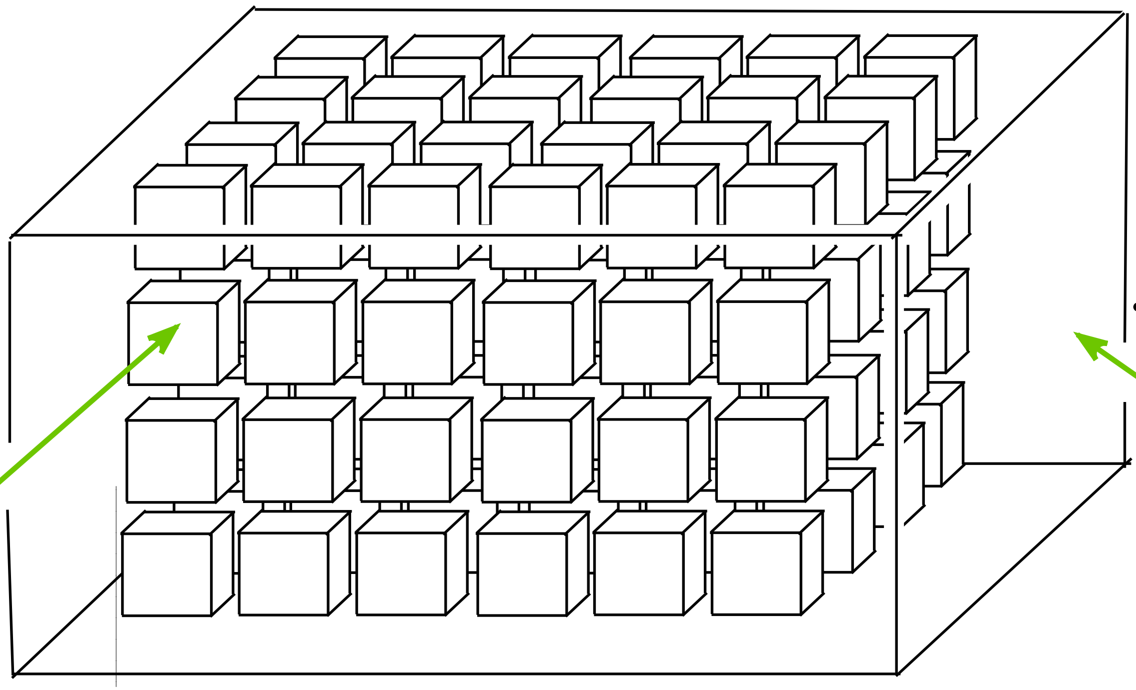
$$\nu_t(x + t\mathbf{v}(x)) = \nu(x\,;\,t\,(\mathrm{div}\,\mathbf{v})(x))$$

Call $\nabla\nu$ and $\partial\nu$ the two partial derivatives

Now $$f = |B|^2/2\,\nabla\nu + \nabla(|B|^2/2\,\partial\nu)$$

A case when "$\partial\nu$" counts

DC coils...

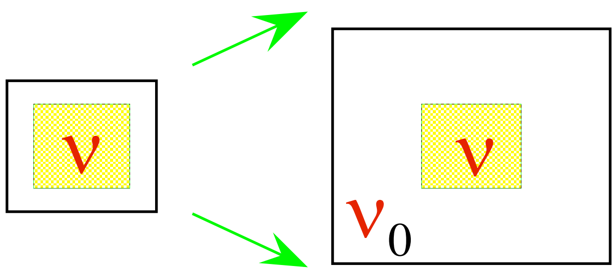


Hard, high μ, material

... create ambient field **B**

Soft, compressive substrate

Homogenized reluctivity ν_{eff}, increases with expansion.

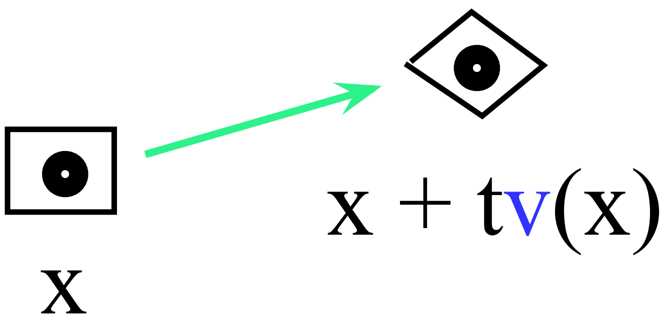


Computing $-\partial_u \Psi$ for anisotropic ν

$$f = |B|^2/2 \, \nabla\nu \quad \rightarrow \quad f = 1/2 \, B^i B^j \, \nabla\nu^{ij}$$

$$\rightarrow \quad f = 1/2 \, (\nabla\nu \, B) \cdot B$$

But again, $\nu_t(x + t\mathrm{v}(x)) = \nu(x)$ not realistic



rotation $t\omega$,

$$\omega = \frac{1}{2} \operatorname{rot} \mathrm{v}$$

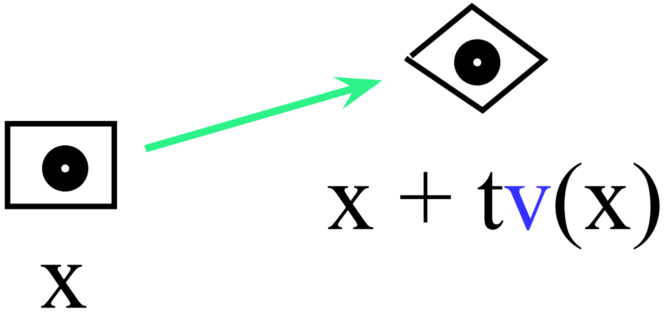
reluctivity follows matter:

$\nu \rightarrow \nu_r$ by rotation r

Claim: $\boxed{\nu_r r = r\nu}$

"Material frame indifference" principle

" (...) souvent évoqué, mais jamais formulé exactement" (J.-M. Souriau)



$$\boxed{v_r(rb)\cdot rb = vb\cdot b} \quad \text{whatever } b$$

rotation $t\omega$,

$$\omega = \frac{1}{2}\,\mathrm{rot}\ v$$

$$(r^{-1} v_r\, r)b\cdot b = vb\cdot b \quad \text{whatever } b$$

$$\Rightarrow \quad v_r r = r v$$

Here, r approximated by $b \rightarrow b + t\omega \times b$

$$\boxed{v_t(x+tv) = v(x) + t[(\omega\times v)(x) - (v\omega)(x)\times]}$$

Computing $-\partial_u \Psi$ for anisotropic ν

$$\nu_t(x + t v) = \nu(x) + t[(\omega \times \nu)(x) - (\nu\omega)(x) \times]$$

Differentiate in t at $t = 0$

$$-\int B \cdot (\partial_t \nu_t|_{t=0} \cdot B) = \int v \cdot (\nabla\nu \cdot B) \cdot B - \int B \cdot (\omega \times \nu - \nu\omega \times)B$$

$$\omega = \frac{1}{2} \operatorname{rot} v \quad \Rightarrow \quad 2\int \omega \cdot (H \times B) = \int v \cdot \operatorname{rot}(H \times B)$$

$$f = \frac{1}{2}(\nabla\nu\, B) \cdot B - \frac{1}{2} \operatorname{rot}(H \times B)$$

Generalizing to convex magnetic energy

So far, in reference state ($u = 0$),

$$\psi(x, b) = \nu(x)|b|^2/2 \ \text{ and } \ \Psi(0, A) = \int dx\ \psi(x, \mathrm{rot}\, A(x))$$

Substitute $\psi(x, b)$ convex in b, p.w. smooth in x. Then,

$$\Psi(t v,\ A) = \int dx\ \psi_t(x, \mathrm{rot}\ A(x))$$

with $\psi_t(x + t v(x), b + t\omega \times b) = \psi(x, b)$. By same tricks,

$$\boxed{f = \nabla\psi(B) - \frac{1}{2}\mathrm{rot}(H \times B)}$$

where $\nabla\psi(B)$ stands for the field $x \to \nabla_x\psi(x, B(x))$

The case of hard magnets: $B = \mu_0 H + M$

$$\psi(x, b) = \mu_0^{-1}(|b|^2/2 - b \cdot M(x))$$

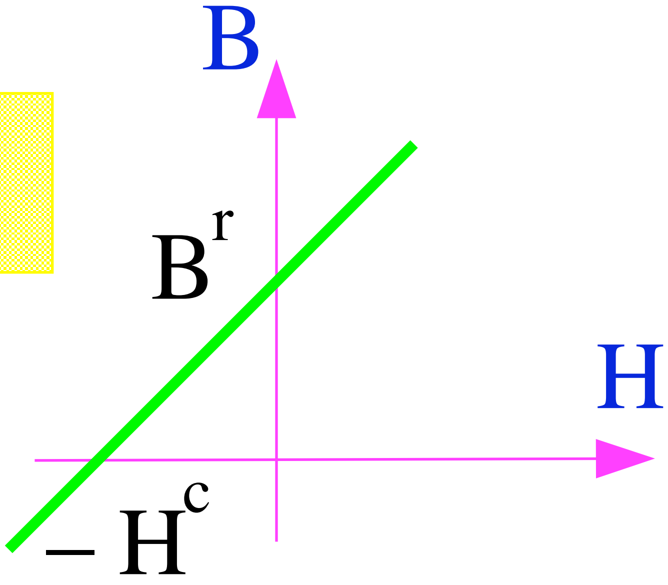
Use previous result to get

$$f = -[\nabla M \cdot B + \frac{1}{2} \mathrm{rot}(B \times M)]/\mu_0$$

but don't forget limited scope of this result: No dependence of M on strain has been assumed

$$B = B^r + \mu_0 H$$

Invariable fluxes of B^r



$$B = \mu_0 (H + H^c)$$

Invariable circulations of H^c

Material seen as aggregation of (permanent) magnetic dipoles.

Material seen as aggregation of (steady) current loops.

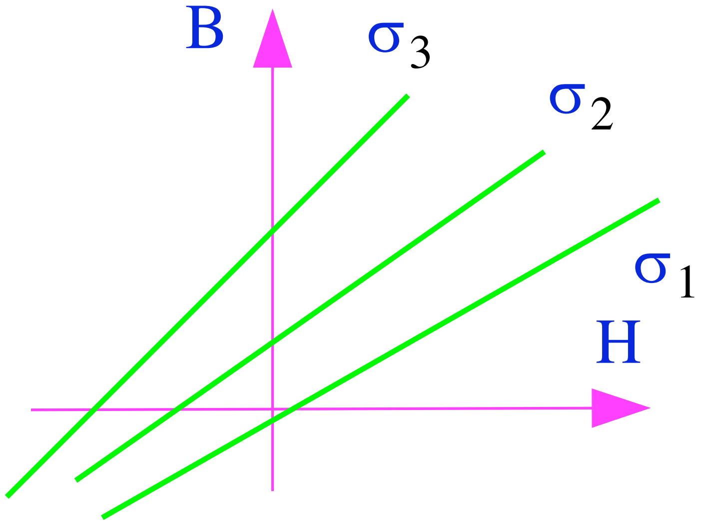
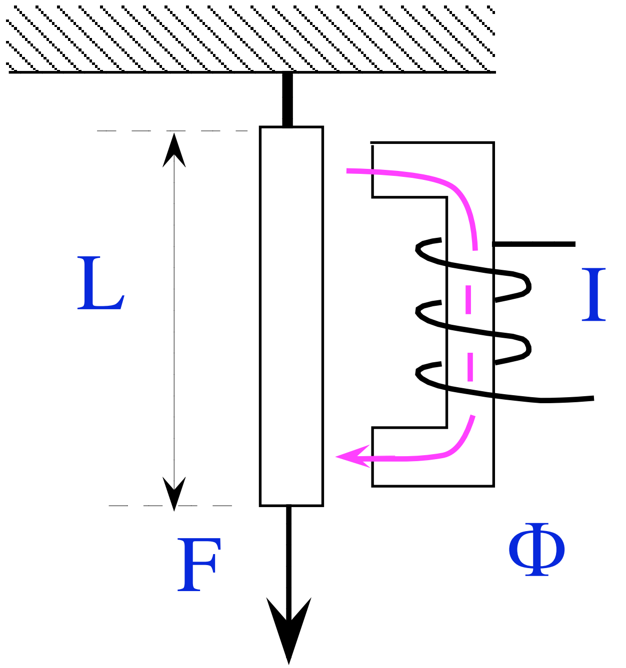
Different force fields, then:

$$\boxed{\begin{aligned} F = J \times B - (H + H^c) \operatorname{div} B^r \\ - (J + J^c) \times B^r \end{aligned}}$$

$$\boxed{F = (J + J^c) \times B}$$

where $J^c = \operatorname{rot} H^c$

Constitutive laws ought to be measured, not theorized about

microscopic level	mesoscopic level	macroscopic level
some lower level of description, e.g., condensed matter physics, etc.		
ψ, etc.	$B, H, \varepsilon, \sigma$	Φ, I, L, F
quantum, etc., laws	local laws	global laws

theory

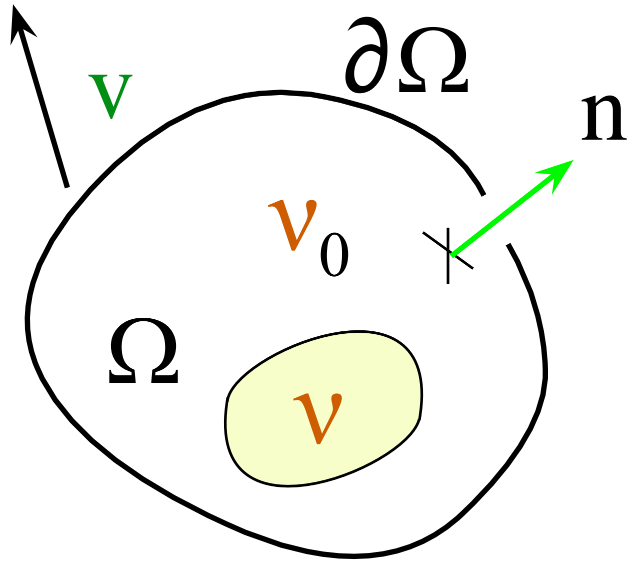
num. simulation

measurement

Merci

Maxwell stress tensor

- div $\mathbf{B} = 0$, $\quad \mathbf{H} = \nu \mathbf{B}$, $\quad$ rot $\mathbf{H} = \mathbf{J}$
- Scalar reluctivity ν (inverse of μ)
  (or if a tensor, $\nu^{ji} = \nu^{ij}$)
- Uniform vector field v



$$\operatorname{div}\left(v \cdot H\, B - \frac{1}{2} B \cdot H\, v\right) = v \cdot \left[\operatorname{rot} H \times B + \frac{1}{2} |B|^2 \nabla \nu\right]$$

$$\int_{\partial\Omega} \left[H^i B^j - \frac{1}{2} B^k H^k \delta_{ij}\right] n^j = \int_{\Omega} \left[\operatorname{rot} H \times B + \frac{1}{2} |B|^2 \nabla \nu\right]^i$$

$$\mathbf{M} = \mathbf{H} \otimes \mathbf{B} - \frac{1}{2} (\mathbf{B} \cdot \mathbf{H})\, \delta$$